An Algebra for Grammar Schools

HOBBS

AN

ALGEBRA

FOR

GRAMMAR SCHOOLS

BY

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PREFACE.

IT is not necessary to make any long explanation as to the desirability of including Algebra in the list of Grammar School studies. The advantages have been fully recognized by the leading educators of our country, and experience has abundantly proved that these advantages are real, and not merely thoretical. In the first place, pupils are enabled to anticipate some of the work formerly assigned to the High School course. Furthermore, many of the stumbling blocks of Arithmetic are rendered quite easy by applying the simple principles of Algebra.

At the outset, the pupils should learn the notation of letters as used in Algebra, and the first chapter of this book leads up to this notation through the use of some simple examples. It is very important that the pupil should realize that the fundamental principles are the same as those that have been learned in Arithmetic.

Equations in their various forms furnish a working basis for the simplification of many of the difficulties of Arithmetic. Accordingly, the next four chapters are devoted to the treatment of equations and their applications to problems. In Chapter IV. the various problems of percentage and interest are considered; not only are these problems solved more easily by the use of formulas,

but in addition the pupil learns methods which are useful in many kinds of practical problems. In Chapter V. ratio and proportion are shown to be only special applications of equations. After a study of the processes of Algebra, equations are still further considered in Chapters XI., XV., and XVII.

Chapter VI. introduces the pupil to negative quantities, and emphasis is laid upon the fact that negative quantites are just as real as positive quantities, and that they differ only in being opposite to each other.

The remaining chapters take up the parts of Algebra that can be studied to advantage in Grammar Schools. It will be found that this book furnishes sufficient material to undertake practical applications in after life. Great care has been taken in the selection of examples in order to avoid those which would be over the heads of Grammar School pupils. At the same time, an abundance of examples has been given in every case sufficient to enable a pupil to obtain a firm grasp on the subject under consideration.

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GRAMMAR SCHOOL ALGEBRA.

CHAPTER I.

INTRODUCTION.

1. The fundamental principles of Mathematics are unchangeable. Many of the same principles and processes are found in Algebra that have previously been learned in Arithmetic. The signs $+$, $-$, $\times$, $\div$, and $=$ are used with exactly the same meanings.

In order to introduce the pupil to the notation of letters, which is the new part of Algebra, a few simple illustrations are here given.

I. Find the sum of 8 apples and 6 apples.

8 apples + 6 apples = 14 apples. If we use the letter a to designate apples, the example becomes $8a + 6a = 14a$. That is, the sum of $8a$ and $6a$ is $14a$.

In the same way it can be shown that the difference between $8a$ and $6a$ is $2a$.

II. Find the sum of 8 apples and 6 bananas.

Using a for apples and b for bananas, the result is $8a + 6b$. That is, the sum of $8a$ and $6b$ is $8a + 6b$; since a and b are not of the same kind, the result cannot be written in any simpler form.

In the same way it can be shown that the difference between $8a$ and $6b$ is $8a - 6b$.

III. Find the area of a rectangular field a rods long and b rods wide.

The area is found by multiplying the length by the breadth. The result is written ab square rods. It is customary to omit the sign $\times$, and thus denote the product by writing one letter after the other.

IV. Find the area of a square field which measures a rods on a side.

Using the method of the preceding example, the result is aa square rods; this may be expressed more briefly a^2 square rods. a^2 denotes the product of two numbers each equal to a.

In like manner, the volume of a cube which measures a feet on an edge is a^3 cubic feet.

V. If a man earns a dollars in b days, how much does he earn in one day?

$a \div b$ is commonly written $\frac{a}{b}$; hence the answer is $\frac{a}{b}$ days.

2. Letters as used in Algebra are called **quantities.** The first and middle letters of the alphabet, as a, b, m, n, are used to denote values supposed to be known; they are called **known quantities.** The last letters of the alphabet, as x, y, z, are used to denote values which are unknown and must be determined by the solution of a problem; they are called **unknown quantities.**

NOTE. The Arabic numerals are known quantities.

A known quantity placed before another quantity to show how many times that quantity is taken is called a **coefficient.** For example, in $8a$, 8 is the coefficient of a.

In $8\,ax$, 8 may be considered the coefficient of ax, or $8\,a$ may be considered the coefficient of x.

NOTE. **The numerical coefficient 1 is usually omitted, and $1a$ is usually written a.**

When several quantities are multiplied together, the result is called the **product**, and each separate quantity is called a **factor** of the product; thus, 8, a, and x are the factors of $8\,ax$. Factors expressed by numbers are called **numerical factors**; those expressed by letters are called **literal factors.**

The product obtained by using a quantity any number of times as a factor is called a **power** of the quantity. The number of times the quantity is to be used as a factor is denoted by a small figure or letter, called an **exponent**, or **index**, placed a little above and at the right of the quantity.

NOTE. **a^2 is read *a second*, or *a square*; a^3 is read *a third*, or *a cube*; a^4 is read *a fourth*; and so on. When no exponent is given, the exponent 1 is understood.**

If a quantity is the product of any number of equal factors, one of these factors is called a **root** of the quantity. If there are two equal factors, the root is called the *square root*, and is indicated by the **radical sign,** $\sqrt{\ }$; thus, $\sqrt{a}$ denotes the *square root* of a. To indicate any other root, a little figure is placed in the opening of the sign; thus, $\sqrt[3]{a}$ denotes the *third root* or *cube root* of a.

3. Any collection of algebraic symbols is called an **algebraic expression.**

An algebraic expression in which the symbols are not connected by the sign $+$ or $-$ is called a **term**; as $5\,a^2b$, or $3\,abc \div 4\,x$.

An algebraic expression containing only one term is called a **simple expression, or monomial.**

An algebraic expression containing two or more terms is called a **compound expression**, or **polynomial**. A polynomial of two terms is called a **binomial**, and a polynomial of three terms is called a **trinomial**. For example, $8a - 6b$ is a binomial; $8a - 6b + c^2$ is a trinomial.

Terms preceded by the sign $+$ are called **positive terms**; terms preceded by the sign $-$ are called **negative terms.**

NOTE. If a monomial or the first term of a polynomial is positive, the sign $+$ is usually omitted before it.

When terms contain the same letters, and each letter has the same exponent in every term, they are called **similar terms**, or **like terms**; thus, $5a^2bc^3$, a^2bc^3, and $-3a^2bc^3$ are similar terms. When terms contain different letters, or the same letters with different exponents, they are called **dissimilar terms**, or **unlike terms**; thus, $3ab$ and $4ac$, and $3a^2b$ and $4a^3b^2$, are pairs of dissimilar terms.

The number of literal factors of a term is called the **degree** of the term. For example, $3a$ is of the first degree; $4abc$ is of the third degree; $5a^2bc^3$, that is, $5aabccc$, is of the sixth degree.

The degree of a polynomial is the same as the degree of its term which is of the highest degree.

A polynomial is said to be *arranged according to the descending powers* of any letter when the first term contains the highest power of that letter, the second term contains the next highest power, and so on; any terms not containing that letter are put last. If the terms are arranged in the reverse order, they are said to be *arranged according to the ascending powers* of the letter. For example, the polynomial $x^3 - 2x^2y - 5xy^2 + 4y^3$ is arranged according to the descending powers of x and according to the ascending powers of y.

4. When a compound expression is used as a whole, it is enclosed in ***parentheses.*** For example, $a-(b+c)$ indicates that the sum of b and c is to be subtracted from a. ***Brackets, braces,*** and the ***vinculum*** are also used in the same manner; thus, $a-[b+c]$, $a-\{b+c\}$, and $a-\overline{b+c}$. These signs are known as the ***signs of aggregation.***

I. Simplify $9+(6+2)$.

This means that $6+2$ is to be added to 9. When 6 is added to 9, the sum is $9+6$, or 15. When a number 2 more than 6 is added to 9, the sum is 2 more than before, and the result is $9+6+2$, or 17. Hence $9+(6+2)=9+6+2$.

II. Simplify $9+(6-2)$.

The result is 2 less than when 6 is added to 9. Hence $9+(6-2)=9+6-2$.

III. Simplify $9-(6+2)$.

This means that $6+2$ is to be subtracted from 9. If 6 is subtracted from 9, the remainder is $9-6$, or 3. When a number 2 more than 6 is subtracted from 9, the remainder is 2 less than before, and the result is $9-6-2$, or 1. Hence $9-(6+2)=9-6-2$.

IV. Simplify $9-(6-2)$.

The result is 2 more than when 6 is subtracted from 9. Hence $9-(6-2)=9-6+2$.

Letting a, b, and c represent any three numbers, a similar investigation gives the following results:

$$\begin{aligned}
a+(b+c)&=a+b+c,\\
a+(b-c)&=a+b-c,\\
a-(b+c)&=a-b-c,\\
a-(b-c)&=a-b+c.
\end{aligned}$$

The principles for the removal of parentheses may be stated as follows:

When parentheses preceded by the sign + are removed, the signs of the terms within the parentheses remain unchanged. When parentheses preceded by the sign − are removed, the sign of every term within the parentheses is changed.

Conversely, *any number of terms may be enclosed in parentheses preceded by the sign + without any change of signs; any number of terms may be enclosed in parentheses preceded by the sign −, provided the sign of every term thus enclosed is changed.*

NOTE. When the first term of an expression within parentheses has no sign before it, the sign + is understood.

V. Simplify $3\,(6+4)$.

This means that $6+4$ is to be multiplied by 3. When $6+4$, or 10, is multiplied by 3, the product is 30. If, however, we multiply 6 and 4 separately by 3, and add the products, we have $3\times 6+3\times 4 = 18+12 = 30$.

Since these two results are the same,

$$3\,(6+4) = 3\times 6+3\times 4.$$

VI. Simplify $3\,(6-4)$.

Using the same method of investigation as in the preceding example, it may be shown that

$$3\,(6-4) = 3\times 6-3\times 4.$$

Using any numbers whatever, the principle remains the same. Hence $a\,(b+c) = ab+ac$

and $a\,(b-c) = ab-ac$

Since the order of the factors is of no consequence,

$$(b+c)\,a = ab+ac,$$

and $(b-c)\,a = ab-ac$

5. To find the numerical value of an algebraic expression when each letter has a particular value, it is necessary to substitute for each letter its particular value, and then perform the operations indicated. In compound expressions, care must be taken to simplify each term by itself before performing the operations of addition and subtraction.

I. If $a = 4$, $b = 2$, and $c = 1$, find the numerical value of (i), $3\,ac$; (ii), $5\,ab^3c^2$.

(i) $3\,ac = 3 \times 4 \times 1 = 12.$

(ii) $5\,ab^3c^2 = 5 \times 4 \times 2^3 \times 1^2 = 5 \times 4 \times 8 \times 1 = 160.$

II. If $a = 5$, $b = 3$, $c = 1$, and $d = 0$, find the numerical value of $ab^2 + 6\,acd - (2\,b^3 - 4\,a)$.

$$\begin{aligned} ab^2 + 6acd - (2b^3 - 4a) &= 5\times3^2 + 6\times5\times1\times0 - (2\times3^3 - 4\times5) \\ &= 5\times9 + 6\times5\times1\times0 - (2\times27 - 4\times5) \\ &= 45 + 0 - (54 - 20) \\ &= 45 + 0 - 34 = 11. \end{aligned}$$

EXAMPLES.

If $a = 1$, $b = 2$, $c = 3$, $d = 4$, and $e = 0$, find the numerical value of

1. $8\,ac$. **3.** $7\,b^3$. **5.** $7\,bc^2e$. **7.** $\frac{1}{8}\,b^2c^3d$.

2. $10\,acd$. **4** $4\,a^4c^2d$. **6.** $2\,a^2b^2c^2d^2$. **8.** $\frac{1}{9}\,a^4c^4e^4$.

9. $a + b + c$.

10. $5\,a - b - c$.

11. $ab + 3\,c$.

12. $8\,a^2b^2 - 2\,c^2$.

13. $a^2 + (b^2 + c^2)$.

14. $ac - (3\,b - d)$.

15. $b\,(c + d)$.

16. $3\,a\,(2\,b - c)$.

17. $8\,a - 3\,(2\,c - d)$.

18. $6\,b^2 - e\,(a^2 + b^2)$.

19. $8\,bc - b\,(c - a)$.

20. $(b - a)^3 + (b + a)^2$.

EXERCISES IN ALGEBRAIC NOTATION.

6. It is essential that the beginner should learn as soon as possible to express the conditions of a problem in algebraic language. If any difficulty arises, it is well to consider what operation would be used in a similar arithmetical problem having numbers in place of letters; then apply the same operation to the notation of letters that would be used with numbers.

1. a is how much more than b?

2. What must be added to c to make d?

3. What number is 10 less than x?

4. By how much does 60 exceed x?

5. By what must 4 be multiplied to obtain y?

6. By what must 6 be divided to obtain z?

7. If one part of 21 is x, what is the other part?

8. If the sum of two numbers is s, and a is one of them, what is the other?

9. If the difference of two numbers is d, and a is the smaller number, what is the larger number?

10. If the difference of two numbers is d, and b is the larger number, what is the smaller number?

11. The product of two numbers is m; if x is one of the numbers, what is the other?

12. If d is the divisor, q is the quotient, and r is the remainder, what is the dividend?

13. Robert has x marbles, and William has 10 more than Robert; how many has William?

14. A boy spends a cents for a pear and x cents for an orange; how much money does he spend?

15. A man bought a pounds of sugar, b pounds of coffee, and c pounds of tea; how many pounds of groceries did he buy?

16. If John has $2x$ cents, and James has $5x$ cents, how many cents have both together? James has how many cents more than John?

17. A farmer, who owned 50 acres of land, sold $3x$ acres. How many acres did he then own?

18. A man earns m dollars a day and spends p dollars a week. How much does he save in two weeks?

19. If x is an odd integral number, what is the next larger odd integral number? The next larger integral number?

20. Find the perimeter of a rectangular field a rods long and b rods wide.

21. How many cents are there in x dollars and y dimes?

22. If I spend d dimes from a purse containing h half-dollars and q quarter-dollars, how many cents have I left?

23. If a yard of cloth costs x dollars, what is the cost of 5 yards? Of c yards?

24. Find the cost in dollars of b books at c cents each.

25. If n apples cost c cents, find the cost of p apples.

26. If oranges are worth d cents a dozen, how much are x oranges worth?

27. A man bought a acres of land for b dollars, and sold it so as to gain c dollars per acre. Find the selling price.

28. If Henry is 12 years old, how old was he x years ago? How old will he be y years hence?

29. A man was 30 years old z years ago; what is his present age?

30. A man is x years old and in m years he will be twice as old as his son. How old will his son be then?

31. If a man walks m miles an hour, how far will he walk in c hours?

32. If a man walks n miles an hour, how long will it take him to walk p miles?

33. If k men can build a wall in h days, how long will it take one man to build it?

34. If a man can do a certain piece of work in a days, what part of it can he do in one day? In c days?

35. A man bought a horse for x dollars, and sold it at a gain of 10%. Find the selling price.

36. In trying to form a regiment into a solid square with x men on a side, it was found that there were 12 men over. How many men are there in the regiment?

CHAPTER II.

SIMPLE EQUATIONS.

7. An expression of equality between two algebraic expressions is called an **equation.** Every equation is made up of two parts; that which precedes the sign of equality is called the **first member,** and that which follows the sign of equality is called the **second member.** For example, in the equation $4x + 5 = 17$, $4x + 5$ is the first member, and 17 is the second member.

Equations are of two kinds: — identical equations and equations of condition. An **identical equation,** or **identity,** is one in which the expression of equality is true, no matter what values are given to the letters; thus, the equation $a + b = b + a$ is true for any values that may be given to a and b. An **equation of condition** is one which is true only for certain values of the letters; thus, the equation $4x + 5 = 17$ is true only when $x = 3$.

NOTE. Whenever the word *equation* is used alone, an equation of condition is meant.

The common use of equations in Algebra is to obtain the values of the unknown quantities.

An equation containing only the first power of x (or other unknown quantity) is called a **simple equation,** or an **equation of the first degree.**

The operation of obtaining the value of the unknown quantity is called the **solution** of the equation, and the value of the unknown quantity is called the **root** of the equation. If the root is substituted for the unknown quantity, the equation becomes an identity, and the **root** is said to **satisfy** the equation.

8. The solution of equations depends upon certain self-evident truths, called **axioms.**

1. Things equal to the same thing are equal to each other.

2. If equals are added to equals, the sums are equal.

3. If equals are subtracted from equals, the remainders are equal.

4. If equals are multiplied by equals, the products are equal.

5. If equals are divided by equals, the quotients are equal.

6. Like powers or like roots of equals are equal.

The principle of **transposition of terms** is founded upon Axioms 2 and 3.

Let $$x + a = b.$$

If the same quantity a is subtracted from both members of this equation, the remainders are equal, according to Axiom 3, and we have

$$x + a - a = b - a.$$

That is, $$x = b - a.$$

Again, let $$x - a = b.$$

If the same quantity a is added to both members of this equation, the sums are equal, according to Axiom 2, and we have

$$x - a + a = b + a.$$

That is, $$x = b + a.$$

In both cases we see that when a is removed from one member of the equation, it appears in the other member with its sign changed. It is evident that the principle

applies to any term of an equation. Hence, *any term may be transposed from one side of an equation to the other, provided its sign is changed.*

If we transpose all the terms of the equation

$$a - x = b - c,$$

we have $$c - b = x - a.$$

This is the same as $x - a = c - b$.

Hence, *the signs of all the terms of an equation may be changed without altering the equality.*

Note. If a term occurs in both members of an equation with the same sign, it may be removed from both without altering the equality.

SOLUTION OF EQUATIONS.

9. I. Solve $5x - 4 = 3x + 6$.

Transposing the terms -4 and $3x$,

$$5x - 3x = 6 + 4.$$

Uniting similar terms, $2x = 10$.

Dividing both members by 2, $x = 5$.

Note. In any equation the terms must be transposed so that all the terms containing x shall be in the first member, and the other terms in the second member.

II. Solve $6x + 8 = 13x - 6$.

Transposing, $6x - 13x = -6 - 8$.

Changing the signs of all the terms,

$$13x - 6x = 6 + 8.$$

Uniting similar terms, $7x = 14$.

Dividing both members by 7, $x = 2$.

III. Solve $8x - 11 - 3(x + 3) = 2(x - 4)$.

Removing parentheses, $8x - 11 - 3x - 9 = 2x - 8$.

Transposing, $8x - 3x - 2x = 11 + 9 - 8$.

Uniting similar terms, $3x = 12$.

Dividing both members by 3, $x = 4$.

If an equation becomes an identity when the value of x is substituted for x, the root is said to be verified.

The verification of the three illustrative problems is as follows:

I. $5x - 4 = 3x + 6.$

When $x = 5$, $5 \times 5 - 4 = 3 \times 5 + 6$
$$25 - 4 = 15 + 6.$$
$$21 = 21.$$

II. $6x + 8 = 13x - 6.$

When $x = 2$, $6 \times 2 + 8 = 13 \times 2 - 6.$
$$12 + 8 = 26 - 6.$$
$$20 = 20.$$

III. $8x - 11 - 3(x + 3) = 2(x - 4).$

When $x = 4$, $8 \times 4 - 11 - 3(4 + 3) = 2(4 - 4).$
$$8 \times 4 - 11 - 3 \times 7 = 2 \times 0.$$
$$32 - 11 - 21 = 0.$$
$$0 = 0.$$

EXAMPLES.

Solve the following equations:

1. $x + 4 = 11.$
2. $x + 3x = 12.$
3. $5x - 4 = 21.$
4. $6x - x = 30.$
5. $3x + 4 = 2x + 7.$
6. $5x + 9 = 3x + 15.$
7. $4x - 3 = x + 9.$
8. $4x - 9 = 19 - 3x.$
9. $3 - 8x = 60 - 11x.$
10. $2x - 11 = -7x + 34.$
11. $x + 4x - 8 = 7.$
12. $3(2x - 3) = 7x - 15.$
13. $4(x - 3) = 3(x - 4).$
14. $2x = 13 - (2x - 3).$
15. $2(x - 6) = -3(x - 1).$
16. $3x - (5x - 6) + 10 = 0.$
17. $3x + 19 - 8(12 - x) = 0.$
18. $3x + 2x - 6 = x + 14.$
19. $6x - 3 - 3x = 2x + 7.$
20. $2x - (3 - x) = 7(x - 1).$

21. $6 - 3x - (1 - 5x) = 26 - x.$

22. $3(x + 5) + 7(x + 1) = 8(x + 4).$

23. $2(x - 2) - 3(x - 3) + 4(x - 4) + 8 = 0.$

24. $x - 7(3x - 5) = 2(x + 1) - 13(9 - x) - 60.$

25. $8x - 4(5x + 3) + 8(7 - 5x) + 476 = 0.$

26. $5x - 11 - 8x + x = 4x - 10 + 2x - 5x - 13.$

27. $x^2 + 6x - 9 = x^2 - 7x + 17.$

28. $x(x + 5) - 10 = x(x - 1) + 8.$

29. $x^2 - 6x + 21 = x(x - 2) - 5(x - 9) - 20.$

30. $x^2 - 3x - (x^2 + 4x - 3) = 22 - 5(x + 7).$

PROBLEMS LEADING TO SIMPLE EQUATIONS.

10. Many arithmetical problems can be solved more easily by the use of equations than by the ordinary processes of Arithmetic. Denoting the required number by x, it is necessary in each problem to obtain from the given conditions two different expressions which represent the same thing. By making these two expressions equal, we have an equation, the solution of which gives the answer to the problem. If two or more numbers are required in any problem, let one of them be denoted by x or some multiple of x, and express the others in terms of x.

I. Edward has four times as many books as Harold, and together they have 60. How many has each?

$x =$ the number of books Harold has.
$4x =$ the number of books Edward has.

$$
\begin{aligned}
x + 4x &= 60. \\
5x &= 60. \\
x &= 12. \\
4x &= 48.
\end{aligned}
$$

Ans. $\begin{cases} \text{Edward, 48 books.} \\ \text{Harold, 12 books.} \end{cases}$

II. The difference of two numbers is 12, and their sum is 30. Find the numbers.

$$\begin{aligned} x &= \text{the smaller number.} \\ x+12 &= \text{the larger number.} \\ x+x+12 &= 30. \\ x+x &= 30-12. \\ 2x &= 18. \\ x &= 9. \\ x+12 &= 21. \qquad \textit{Ans. } 9 \text{ and } 21. \end{aligned}$$

Another solution.

$$\begin{aligned} x &= \text{the larger number.} \\ x-12 &= \text{the smaller number.} \\ x+x-12 &= 30. \\ x+x &= 30+12. \\ 2x &= 42. \\ x &= 21. \\ x-12 &= 9. \qquad \textit{Ans. } 9 \text{ and } 21. \end{aligned}$$

III. The sum of two numbers is 44, and the larger is 5 more than twice the smaller. Find the numbers.

$$\begin{aligned} x &= \text{the smaller number.} \\ 2x+5 &= \text{the larger number.} \\ x+2x+5 &= 44. \\ x+2x &= 44-5. \\ 3x &= 39. \\ x &= 13. \\ 2x+5 &= 31. \qquad \textit{Ans. } 13 \text{ and } 31. \end{aligned}$$

IV. A has \$580, and B has \$140. How many dollars must A give to B in order that he may have just twice as much as B?

$$\begin{aligned} x &= \text{the number of dollars A must give to B.} \\ 580-x &= 2\,(140+x). \\ 580-x &= 280+2x. \\ -x-2x &= 280-580. \\ x+2x &= 580-280. \\ 3x &= 300. \\ x &= 100. \qquad \textit{Ans. } \$100. \end{aligned}$$

V. A is now four times as old as B, but in 6 years he will be only three times as old. Find the age of each.

$$x = \text{the number of years in B's age.}$$
$$4x = \text{the number of years in A's age.}$$
$$4x + 6 = 3(x + 6).$$
$$4x + 6 = 3x + 18.$$
$$4x - 3x = 18 - 6.$$
$$x = 12.$$
$$4x = 48.$$

Ans. $\begin{cases} \text{A, 48 years.} \\ \text{B, 12 years.} \end{cases}$

NOTE. Avoid letting x stand for such vague expressions as money, time, distance, etc. When the answer is to be a concrete number, always let x represent a number of units of some kind, as dollars, years, miles, etc. In any problem, quantities of the same kind must be expressed in terms of the same unit.

EXAMPLES.

1. What number must be added four times to itself in order that the sum may be 120?

2. The sum of two numbers is 315, and the greater is six times the less. Find the numbers.

3. The difference of two numbers is 66, and the greater is four times the less. Find the numbers.

4. The larger of two numbers is 9 more than the smaller, and their sum is 31. What are the numbers?

5. John is three times as old as Henry, and the sum of their ages is 24 years. What is the age of each?

6. How can $450 be divided between two men so that one shall have eight times as much as the other?

7. If to a certain number you add seven times the number, the sum is 36 more than five times the number. Find the number.

8. The sum of two numbers is 19 and their difference is 5. What are the numbers?

9. The difference between two numbers is 15, and four times the greater is equal to seven times the less. What are the numbers?

10. Find two numbers differing by 8 such that their sum shall be seven times their difference.

11. Find two numbers differing by 20 such that five times the smaller shall be 7 less than twice the larger.

12. A man had $128, and spent part of it; he then had three times as much as he had spent. How much did he spend?

13. Mary's mother gave her some money, and her father gave her twice as much. After spending 25 cents, she had 95 cents left. How much did she receive from each parent?

14. Carl has 4 more than twice as many marbles as Robert, and both together have 58. How many has each?

15. A has $580, and B has $390. How many dollars must A give to B in order that each may have the same amount?

16. A has $760, and B has $220. How many dollars must A give to B in order that he may have just three times as much as B?

17. A and B invest equal amounts in business. A gains $840 and B loses $330; then A has twice as much as B. What sum did each invest?

18. A and B had the same amount of money. A gave B $60, and then B had twice as much as A. How much did each have at first?

19. Two shepherds owned a flock of sheep. When they agreed to dissolve the partnership, A took 60 sheep, and B took 84 sheep and paid A $84. Find the value of one sheep.

20. A man doubled his capital every year for 5 years, and then had $12800. What was his capital at first?

21. A man bought a watch and chain for $90. The watch cost $2 more than ten times the price of the chain. Find the cost of each.

22. A man sold a quantity of eggs at 40 cents a dozen, and twice as many at 45 cents a dozen. The entire amount he received was $26. How many eggs of each kind did he sell?

23. A boy bought a bat and a ball for $1.50, paying five times as much for the ball as for the bat. How much did he pay for each?

24. Find three consecutive numbers whose sum is 51.

25. Divide 135 into three parts such that the second shall be 10 more than the first, and the third 10 more than the second.

26. The sum of three numbers is 60. The first is 9 more than the second, and the third is 6 less than the first. Find the numbers.

27. The sum of three numbers is 174. The second is twice the first, and the third is equal to the sum of the other two. Find the numbers.

28. Three brothers have birthdays 2 years apart, and their united ages amount to 48 years. Find their ages.

29. A boy paid 64 cents for an equal number of apples, pears, and oranges. The apples were 1 cent each, the pears 2 cents each, and the oranges 5 cents each. How many of each did he buy?

30. A man paid $20 for a suit of clothes. He paid three times as much for the trousers as for the vest, and twice as much for the coat as for the trousers. How much did he pay for each garment?

31. A man paid a bill of $2.55 in quarters and dimes, using three times as many quarters as dimes. Find the number of coins of each kind.

32. A room is 28 feet long, which is 2 feet more than twice the width. How wide is the room?

33. A rectangular field is twice as long as it is wide, and the perimeter is 72 rods. Find the length and the breadth.

34. A man is three times as old as his son, and his son is 6 years older than his daughter. If the sum of their ages is 64 years, find the age of each.

35. Frank is 16 years old, and he is 2 years less than twice as old as his sister. How old is his sister?

36. A man 36 years old has a son 8 years old. In how many years will the father be twice as old as the son?

37. A man is four times as old as his son, but 4 years hence he will be only three times as old. Find the age of each.

38. A is 44 years older than B, and A's age is as much above 60 years as B's age is below 40 years. Find the age of each.

CHAPTER III.

FRACTIONAL EQUATIONS.

11. I. Solve $\frac{4}{5}x - 6 = \frac{1}{3}x + 8$.

Transposing, $\frac{4}{5}x - \frac{1}{3}x = 8 + 6.$
Uniting similar terms, $\frac{7}{15}x = 14.$
Dividing both members by $\frac{7}{15}$, $x = 30.$

This equation can be solved more simply by using the process known as **clearing of fractions.**

I. Solve $\dfrac{4x}{5} - 6 = \dfrac{x}{3} + 8.$

Multiplying both members by 15,

$$12x - 90 = 5x + 120.$$

Transposing, $12x - 5x = 120 + 90.$
Uniting similar terms, $7x = 210.$
Dividing both members by 7, $x = 30.$

NOTE 1. $\frac{3}{4}x$ and $\frac{1}{3}x$ are the same as $\dfrac{3x}{4}$ and $\dfrac{x}{3}$. The latter method of writing these terms is the more common.

NOTE 2. Always take the least common multiple of the denominators for the multiplier. In multiplying a fractional term, divide the multiplier by the denominator of the fraction and multiply the numerator by the quotient thus obtained.

II. Solve $\dfrac{x+9}{4} - \dfrac{11-4x}{3} = 1 + \dfrac{7x+13}{6}.$

Multiplying both members by 12,

$$3x + 27 - 44 + 16x = 12 + 14x + 26.$$
$$3x + 16x - 14x = 12 + 26 - 27 + 44.$$
$$5x = 55.$$
$$x = 11.$$

Note. The fraction $\frac{11-4x}{3}$ is the same as $\frac{1}{3}(11-4x)$. Hence, if a fraction is preceded by a minus sign, the sign of every term of the numerator must be changed when the denominator is removed in the process of clearing of fractions.

EXAMPLES.

Solve the following equations:

1. $\frac{x}{3}+3x=30.$

2. $2x-\frac{x}{2}=15.$

3. $\frac{x}{5}+\frac{x}{4}=18.$

4. $\frac{x}{4}-\frac{x}{6}=2.$

5. $x+\frac{x}{2}+\frac{x}{3}=22.$

6. $\frac{x}{2}-8=\frac{x}{4}-4.$

7. $\frac{x}{4}+\frac{x}{2}-\frac{x}{5}=33.$

8. $\frac{3x}{5}-7=\frac{7x}{10}-10.$

9. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=19\frac{1}{2}.$

10. $2x+\frac{2x}{3}-\frac{3x}{4}=23.$

11. $x-\frac{x}{4}+\frac{1}{2}=\frac{x}{4}+4.$

12. $\frac{1-2x}{7}=\frac{3-5x}{4}+10.$

13. $\frac{x+1}{3}+\frac{3x-5}{5}=\frac{9x}{10}.$

14. $\frac{x}{4}-\frac{x-1}{11}=x-10.$

15. $3x=\frac{85-x}{2}-\frac{5x-7}{12}.$

16. $\frac{5x+3}{8}-\frac{17+x}{2}=3x-11.$

17. $2x+\frac{5x-5}{4}=12-\frac{5-4x}{5}.$

18. $\frac{x-4}{5}-\frac{x-5}{4}=\frac{x-9}{10}.$

19. $\frac{x-1}{5}-\frac{5x-7}{12}=\frac{16-2x}{3}.$

20. $\frac{x+1}{2}+\frac{x+2}{3}=\frac{x+3}{4}+8.$

21. $\frac{x}{9}+\frac{x-6}{6}=x-\frac{4x-9}{5}.$

22. $\frac{1}{2}(2-x)-\frac{1}{5}(3x+21)=4-2x.$

23. $\frac{1}{4}(3x-1)-\frac{1}{3}(2x+1)+\frac{1}{6}(3x+4)=5.$

24. $\frac{1}{2}(x+1) - \frac{1}{5}(x+2) = \frac{1}{7}(3x-2)$.

25. $\frac{1}{2}(x+3) - \frac{1}{3}(x-2) = \frac{1}{12}(3x-5) + \frac{1}{4}$.

PROBLEMS LEADING TO FRACTIONAL EQUATIONS.

12. I. The difference between two numbers is 6, and $\frac{1}{2}$ of the larger is 9 more than $\frac{1}{5}$ of the smaller. What are the numbers?

$$x = \text{the smaller number.}$$
$$x + 6 = \text{the larger number.}$$
$$\frac{x+6}{2} = \frac{x}{5} + 9.$$
$$5x + 30 = 2x + 90.$$
$$5x - 2x = 90 - 30.$$
$$3x = 60.$$
$$x = 20.$$
$$x + 6 = 26.$$

Ans. 20 and 26.

EXAMPLES.

1. What number must be added to $\frac{3}{4}$ of itself in order that the sum may be 42?

2. If $\frac{1}{4}$ of a number is added to $\frac{1}{5}$ of the same number, the sum is 72. What is the number?

3. What number is that, $\frac{3}{5}$ of which exceeds $\frac{1}{4}$ of it by 21?

4. The sum of two numbers is 56, and the smaller number is $\frac{3}{4}$ of the larger. What are the numbers?

5. The difference between two numbers is 20, and the smaller number is $\frac{5}{9}$ of the larger. What are the numbers?

6. The difference between two numbers is 16; if the larger is divided by the smaller, the quotient is 3. What are the numbers?

7. The sum of two numbers is 84, and $\frac{1}{3}$ of the larger is equal to $\frac{3}{5}$ of the smaller. What are the numbers?

8. If to $\frac{1}{4}$ of a certain number you add $\frac{1}{5}$ of the same number, the sum will be 3 more than $\frac{3}{10}$ of the number. Find the number.

9. If from twice a number I subtract $\frac{1}{3}$ of the number and 5, the remainder is 150. What is the number?

10. The sum of $\frac{3}{8}$ and $\frac{5}{12}$ of a number is 10 less than the number itself. What is the number?

11. The sum of $\frac{1}{6}$ and $\frac{1}{8}$ of a certain number is 2 more than three times the difference between $\frac{1}{3}$ and $\frac{1}{4}$ of the number. What is the number?

12. Divide 70 into two parts such that if one part be divided by 7 and the other by 4, the sum of the quotients shall be 13.

13. Find two consecutive numbers such that $\frac{1}{4}$ of the larger shall be 2 more than $\frac{1}{5}$ of the smaller.

14. A man sold a cow for \$72, and gained $\frac{1}{3}$ of what the cow cost. Find the cost.

15. A boy lost $\frac{3}{7}$ of his money, and had 40 cents left. How much did he have at first?

16. A horse and carriage together cost \$300, and the carriage cost $\frac{2}{3}$ as much as the horse. Find the cost of each.

17. A boy spent $\frac{3}{8}$ of his money one day, and $\frac{7}{12}$ of it the next day, and then had 35 cents left. How much did he have at first?

18. A man paid $\frac{3}{8}$ as much for a cow as for a horse, and the price of the cow added to $\frac{1}{3}$ of the price of the horse was \$85. Find the price of each.

19. John has 9 more than $\frac{1}{2}$ as many cents as Henry, and together they have 57 cents. How many cents has each?

20. Divide \$125 between two men so that $\frac{1}{3}$ of the amount received by the first shall be equal to $\frac{6}{7}$ of the amount received by the second.

21. B has $\frac{3}{4}$ as much money as C, and A has $\frac{5}{6}$ as much as B; together they have \$76. How much has each?

22. A has \$210, and B has \$180. A gives B a certain sum and then has $\frac{4}{9}$ as much as B. What is the sum?

23. A man spends every year \$150 more than $\frac{1}{2}$ of his salary. If he saves \$1200 in 3 years, how much is his salary?

24. How can \$400 be divided among three persons so that the second shall have \$20 more than $\frac{1}{3}$ as much as the first, and the third \$10 more than $\frac{1}{2}$ as much as the second?

25. The length of a rectangular field is $1\frac{3}{4}$ times its breadth, and the perimeter of the field is 88 rods. Find the length and the breadth.

26. The width of a room is $\frac{3}{5}$ of its length. If the length had been 4 feet less, and the width 4 feet more, the room would have been square. Find the length and the breadth.

27. Edith is $\frac{1}{2}$ as old as Sarah, and the sum of their ages is 27 years. Find the age of each.

28. Jane is $\frac{2}{3}$ as old as Mary, and the difference between their ages is 6 years. Find the age of each.

29. Rachel is 3 years older than Helen. If $\frac{3}{4}$ of Helen's age is added to Rachel's age, the sum is 24 years. Find the age of each.

30. George is $\frac{2}{3}$ as old as James, and 8 years ago he was $\frac{2}{5}$ as old. Find the age of each.

CHAPTER IV.

DECIMAL EQUATIONS.

13. I. Solve $.7x - 4 = .417x - .23x + 4.208$.

Transposing, $.7x - .417x + .23x = 4.208 + 4$.
Uniting similar terms, $.513x = 8.208$.
Dividing both members by .513, $x = 16$.

The equation can also be solved by expressing the decimals as common fractions.

I. Solve $.7x - 4 = .417x - .23x + 4.208$.

Expressing the decimals as common fractions,

$$\frac{7x}{10} - 4 = \frac{417x}{1000} - \frac{23x}{100} + \frac{4208}{1000}.$$

Multiplying both members by 1000,

$$700x - 4000 = 417x - 230x + 4208.$$

Transposing, $700x - 417x + 230x = 4208 + 4000$.
Uniting similar terms, $513x = 8208$.
Dividing both members by 513, $x = 16$.

EXAMPLES.

Solve the following equations:

1. $x + .06x = 848$.

2. $20x + 1.2 = 50x$.

3. $.25x - .125x = 1.5$.

4. $.5x = .07x + 1.29$.

5. $2.25x + .125 = 3x - 4$.

6. $.6x - .01x + .005x = 47.6$.

7. $.42x - 3.04 = .09x - 2.842.$

8. $.625x - 2.875 = .14x + 2.945.$

9. $.7x + 4 - .25x = .3x + 7.$

10. $.4x - .4 - .04x = .516x - 1.18.$

PERCENTAGE.

14. I. Find 15% of 240.

$$240 \times .15 = 36.$$

In the above example, 240 is the base, .15 is the rate per cent, and 36 is the percentage. In all such examples, the percentage is equal to the product of the base and the rate per cent. If p, b, and r are used to denote respectively percentage, base, and rate per cent, $p = br$.

NOTE. A rule stated in the form of an equation is called a *formula*.

II. A clerk spends $864 a year, which is 72% of his salary. What is his salary?

$x =$ the number of dollars in the salary.
$p = 864$, $b = x$, and $r = .72$; hence according to the formula $p = br$,

$$864 = x \times .72.$$

Changing the order of the members,

$$.72x = 864.$$
$$x = 1200. \qquad \textit{Ans. } \$1200.$$

III. A man bought a horse for $250, and sold it for $280. What per cent did he gain?

$$x = \text{gain per cent.}$$
$$250 + 250x = 280.$$
$$250x = 280 - 250.$$
$$250x = 30.$$
$$x = .12. \qquad \textit{Ans. } 12\%.$$

IV. A man makes two investments amounting together to \$2000. On one investment his income is at the rate of 6%, and on the other 4%; the total income is \$110. Find the amount of each investment.

$$\begin{aligned}
x &= \text{the number of dollars invested at } 6\%.\\
2000 - x &= \text{the number of dollars invested at } 4\%.\\
.06x + .04(2000 - x) &= 110.\\
.06x + 80 - .04x &= 110.\\
.06x - .04x &= 110 - 80.\\
.02x &= 30.\\
x &= 1500.\\
2000 - x &= 500.
\end{aligned}$$

Ans. $\begin{cases} \$1500 \text{ at } 6\%. \\ \$500 \text{ at } 4\%. \end{cases}$

EXAMPLES.

1. A clerk has a salary of \$1200, and pays 15% of it for house rent. What rent does he pay?

2. A boy's age is 60% less than his father's age, and his father is 40 years old. Find the boy's age.

3. A man weighs 90% more than when as a boy he weighed 80 pounds. What is his weight?

4. If a house worth \$3500 rents for \$280 a year, what per cent of its value is the rent?

5. A clerk has a salary of \$1400, and his expenses for a year amounted to \$1008. What per cent of his salary did he save?

6. A house which cost \$4500 was sold for \$5400. What per cent was gained?

7. In a certain school there are 195 girls, and the number of girls is 60% of the total number of pupils. How many pupils are there in the school?

8. A collector, who charges 8% commission, remits \$713 to his employer. How much did he collect?

9. A boy sold a book for \$1.10, and thereby lost 12%. Find the cost of the book.

10. The sum of two numbers is 85, and one is 30% less than the other. What are the numbers?

11. The difference between two numbers is 30, and one is 25% more than the other. What are the numbers?

12. The ages of two sisters together amount to 32 years, and the age of the younger is 60% of the age of the older. Find the age of each.

13. A and B gain in business \$5040, of which A is to have 10% more than B. What is the share of each?

14. Two farmers together own 495 acres of land, and one farm is 20% smaller than the other. Find the size of each farm.

15. A man bought three horses for \$730. The second horse cost 20% more than the first, and the third horse cost 28% less than the first. What was the cost of each?

16. \$618 includes a sum to be invested and a commission of 3% of the sum to be invested; what is the sum?

17. A commission merchant receives \$1230 to invest after deducting his commission of $2\frac{1}{2}$%. What sum can he invest?

18. A commission merchant received a sum of money to invest after deducting his commission of $2\frac{1}{2}$%. He invested \$1360. What was the sum he received?

19. A man bought two horses at the same price. He sold one at a gain of 25% and the other at a gain of 20%, receiving \$294 for both horses. How much did he pay for them?

20. A man sold a horse for 25% more than he paid for it. The buying and selling prices together amounted to $315. Find the cost.

21. A man invests one-half of a certain sum at 5% and the other half at 4%, and receives an income of $108. Find the sum invested.

22. A man invests one-third of a certain sum at 5% and the remaining two-thirds at $4\frac{1}{2}$%, and receives an income of $140. Find the sum invested.

23. A man loaned $1000, part at 6%, and the rest at 5%. The yearly interest amounted to $57. Find the amount loaned at each rate.

24. A has a certain sum of money invested at 5%, and B has $800 less invested at 6%. A's income is $30 more than B's. Find the amount of each investment.

25. A man invests $1500 in two parts. At the end of a year he finds that he has gained 20% on one investment, but has lost 6% on the other, and that the value of both is $1566. Find the amount of each investment.

INTEREST.

15. In the percentage examples considered in the preceding section, no reference has been made to time. The interest of a sum of money for one year is a simple case of percentage; for any given number of years, the interest for one year must be multiplied by the number of years. If i, p, r, and t are used to denote respectively interest, principal, rate per cent, and time (in years), $i = prt$. The amount is the sum of the principal and the interest; denoting the amount by a, $a = p + i$, or $a = p + prt$.

I. How long must \$1330 be on interest at 7% to gain \$325.85?

$x =$ the number of years in the time.
$i = 325.85$, $p = 1330$, $r = .07$, and $t = x$; hence according to the formula $i = prt$,

$$325.85 = 1330 \times .07 \times x.$$

Performing the multiplication indicated, and changing the order of the members,

$$93.1\,x = 325.85.$$
$$x = 3.5. \qquad \textit{Ans.}\ 3 \text{ years } 6 \text{ months.}$$

II. Find the principal which will amount to \$1675 in 2 years 4 months at 5%.

$x =$ the number of dollars in the principal.
$a = 1675$, $p = x$, $r = .05$, and $t = 2\frac{1}{3}$; hence according to the formula $a = p + prt$,

$$1675 = x + x \times .05 \times 2\tfrac{1}{3}.$$

Performing the multiplication indicated, and changing the order of the members,

$$x + .11\tfrac{2}{3}\,x = 1675.$$
$$3\,x + .35\,x = 5025.$$
$$3.35\,x = 5025.$$
$$x = 1500. \qquad \textit{Ans.}\ \$1500.$$

EXAMPLES.

1. Find the interest of \$850 for 4 years at 5%.

2. Find the interest of \$1275 for 2 years 9 months at 4%.

3. Find the amount of \$1500 for 5 years 6 months at $4\frac{1}{2}$%.

4. At what rate per cent must \$370 be placed on interest to gain \$55.50 in 3 years?

5. Find the rate per cent when the interest of \$720 for 3 years 6 months is \$113.40.

6. At what rate per cent must $760 be loaned to amount to $881.60 in 2 years 8 months?

7. At what rate per cent will $900 amount to $954 in 9 months?

8. At what rate per cent will a sum of money double itself in 12 years 6 months?

9. How long must $180 be on interest at $5\frac{1}{2}$% to gain $99?

10. How long must $908 be on interest at $3\frac{1}{2}$% to gain $79.45?

11. How long must $570 be on interest at 6% to amount to $855?

12. In what time will $1360 at 4% amount to $1455.20?

13. In what time will a sum of money double itself at 6%?

14. What principal will produce $51 interest in 4 years at 6%?

15. What principal will yield an interest of $339.20 in 5 years 4 months at 6%?

16. What principal at 5% will yield $12.50 interest in 6 months?

17. What principal will produce $349.32 interest in 3 years 5 months at 6%?

18. What sum of money put at interest at 7% will amount to $960 in 4 years?

19. What principal will amount to $780 in 3 years 9 months at 8%?

20. What principal will amount to $1312.50 in 8 months at $7\frac{1}{2}$%?

CHAPTER V.

RATIO AND PROPORTION.

16. The relation between two quantities, as expressed by division, is called their **ratio**, and the two quantities are called the **terms** of the ratio.

The sign of ratio is the colon (:), which is the division sign with the line omitted. Ratio may also be expressed in the form of a fraction. For example, the ratio of a to b is written $a:b$ or $\frac{a}{b}$.

When a number is applied to some particular object or objects, it is called a **concrete number**; when not applied to any object, it is called an **abstract number.** For example, 4 and 7 are abstract numbers, but 4 boys and 7 feet are concrete numbers.

In order to write the ratio of two concrete numbers, they should both be expressed in terms of the same unit. Such a ratio is equal to the ratio of the corresponding abstract numbers. For example, 7 feet : 11 feet equals $7:11$.

An expression of equality between two ratios is called a **proportion.** For example, $a:b=c:d$ signifies that the ratio of a to b equals the ratio of c to d. It is read "a is to b as c is to d."

NOTE. This proportion may be written $a:b::c:d$, the sign of equality being replaced by a double colon (::).

The first and fourth terms of a proportion are called the **extremes**, and the second and third terms are called the **means**.

Writing the proportion $a:b=c:d$ in the fractional form,

$$\frac{a}{b}=\frac{c}{d},$$

we see that a proportion is merely a fractional equation.

Clearing of fractions, $ad=bc$.

That is, *in any proportion the product of the extremes equals the product of the means.*

I. Solve the proportion $12:15=x:35$.

Making the product of the means equal to the product of the extremes,

$$15x=420.$$
$$x=28.$$

The numerical part of this example may be simplified by cancellation.

$$15x=12\times 35.$$

$$x=\frac{\overset{4}{\cancel{12}}\times\overset{7}{\cancel{35}}}{\underset{3}{\cancel{15}}}=28.$$

EXAMPLES.

Solve the following proportions:

1. $x:54=20:72.$
2. $50:x=35:63.$
3. $120:84=x:35.$
4. $25:55=20:x.$
5. $72:x=60:15.$
6. $17:37=x:259.$
7. $96:36=56:x.$
8. $x:12=39:13.$
9. $45:x=69:46.$
10. $99:18=132:x.$

PROBLEMS LEADING TO PROPORTIONS.

17. I. If 12 yards of silk cost \$33, how many yards can be bought for \$55?

x = the number of yards bought for \$55.

The cost varies as the number of yards; that is, the ratio of the number of yards equals the ratio of the cost.

$$12:x=33:55.$$
$$33x=660.$$
$$x=20.$$

Ans. **20 yards.**

II. If 8 men can do a piece of work in 15 days, how long will it take 10 men to do the same work?

x = the number of days it will take 10 men.

Since more men can do the work in less time, the time varies inversely as the number of men, and the second ratio must be written in the inverse order of the first.

$$8:10 = x:15.$$
$$10x = 120.$$
$$x = 12.$$ *Ans.* 12 days.

EXAMPLES.

1. If a man earns \$15 in 6 days, how much will he earn in 16 days?

2. If the rent of 36 acres of land is \$84, how many acres can be rented for \$140?

3. If 18 barrels of flour last a garrison 8 weeks, how long will 63 barrels last?

4. If 12 men can do a piece of work in 14 days, how many men will it take to do it in 4 days?

5. When \$120 is paid for 16 barrels of flour, what will 26 barrels cost?

6. If 36 bushels of wheat make 8 barrels of flour, how many bushels of wheat will be required to make 50 barrels?

7. If 8 men can build a wall 14 rods long in a day, how long a wall will 26 men build in the same time?

8. If a train runs 225 miles in 6 hours, how long will it take it to run 150 miles?

9. If an electric car runs 5 miles in 24 minutes, how far will it run in 3 hours?

10. A field can be mowed in 4 days of 10 hours each; how many days of 8 hours each will it take?

11. If 16 men can reap a field in 9 days, how long would it take when 4 men refuse to work?

12. If 30 acres of land produce 480 barrels of wheat, how many acres are necessary to produce 1200 bushels?

13. A man has enough hay to keep 5 horses 4 months. How long will it last if he buys 3 more horses?

14. If a tap discharging 6 gallons a minute empties a cistern in 6 hours, how long will it take a tap discharging 4 gallons a minute to empty it?

15. When a pole 15 feet high casts a shadow 6 feet long, how long a shadow will be cast by a steeple 165 feet high?

16. When a boy 5 feet in height casts a shadow $4\frac{1}{2}$ feet long, what is the height of a tree that casts a shadow 36 feet long?

17. If 25 yards of silk 18 inches wide are required for a dress, how many yards of cloth 30 inches wide would be required for a similar dress?

18. A circle 22 inches in circumference has a diameter of 7 inches. Find the diameter of a tree whose circumference is $16\frac{1}{2}$ feet.

19. If a hill 960 feet high is represented by a drawing 4 inches high, what height on the same scale will represent a mountain 6000 feet high?

20. If the interest of \$1500 is \$60, what will be the interest of \$2200 for the same time at the same rate?

21. If \$240 yields \$45 interest, how much must be invested to yield \$150 in the same time at the same rate?

22. If the interest of \$2000 at $4\frac{1}{2}\%$ is \$135, at what rate per cent will it gain \$180 in the same time?

18. If the ratio of two unknown quantities is represented by $a:b$, the quantities may be expressed by ax and bx.

I. Two numbers have the same ratio as 3 and 5, and their sum is 152. What are the numbers?

$$\begin{aligned} 3x &= \text{the smaller number.} \\ 5x &= \text{the larger number.} \\ 3x + 5x &= 152. \\ 8x &= 152. \\ x &= 19. \\ 3x &= 57. \\ 5x &= 95. \qquad \textit{Ans. } 57 \text{ and } 95. \end{aligned}$$

Another solution.

$$\begin{aligned} x &= \text{the smaller number.} \\ 152 - x &= \text{the larger number.} \\ x : 152 - x &= 3:5. \\ 5x &= 3(152 - x). \\ 5x &= 456 - 3x. \\ 5x + 3x &= 456. \\ 8x &= 456. \\ x &= 57. \\ 152 - x &= 95. \qquad \textit{Ans. } 57 \text{ and } 95. \end{aligned}$$

EXAMPLES.

1. Two numbers have the same ratio as 5 and 8, and their sum is 182. What are the numbers?

2. Two numbers have the same ratio as 4 and 9, and their difference is 35. What are the numbers?

3. Coffee is mixed in the ratio of 2 pounds of Java to 1 pound of Mocha; how much of each kind is there in a mixture weighing 45 pounds?

4. An alloy contains 13 parts of copper to 7 parts of zinc; how much of each metal is contained in 330 pounds of the alloy?

5. A and B hired a pasture for \$56. A put in 6 horses and B put in 10 horses. How much of the rent ought each to pay?

6. A's age is to B's age as 7 to 9, and the difference of their ages is 8 years. Find the age of each.

7. Separate 180 into three parts which shall be to each other as 2, 5, and 8.

8. The sides of a triangle are to each other as 3, 4, and 5, and the perimeter is 240 feet. Find the length of each side.

9. A father divided \$6720 among his three sons in parts proportional to their ages, which were 13 years, 15 years, and 20 years. How much did each receive?

10. Divide 84 into three parts, such that the first is to the second as 3 is to 4, and the third equals the sum of the other two.

11. The sum of three numbers is 325. The first is to the second as 4 is to 7, and the third is twice the second. Find the numbers.

12. Find two numbers in the ratio of 2 to 3, such that when each is increased by 5, they shall be in the ratio of 3 to 4.

13. The ages of two persons are in the ratio of 5 to 6, and 12 years ago they were in the ratio of 3 to 4. Find their ages.

14. Two numbers are to each other as 7 is to 5. If 9 is added to the greater and subtracted from the less, the sum of the resulting numbers is to their difference as 17 is to 7. Find the numbers.

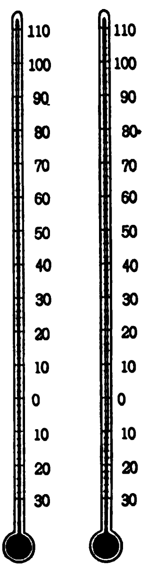
15. What number must be added to each term of the ratio 23 : 29 in order that it may be equal to 5 : 6?

CHAPTER VI.

POSITIVE AND NEGATIVE NUMBERS.

19. In § 3 we have learned that terms preceded by the sign $+$ are called positive terms, and terms preceded by the sign $-$ are called negative terms. In other words, a term which is to be added is a positive term, and a term which is to be subtracted is a negative term. Since addition and subtraction are opposite operations, the signs $+$ and $-$ are opposite in their nature.

In Arithmetic the signs $+$ and $-$ are used to denote addition and subtraction respectively. In Algebra they continue to be used as signs of operation, and they are also used to distinguish quantities opposite in nature or quality. A quantity preceded by the sign $+$ is called a **positive quantity**, and a quantity preceded by the sign $-$ is called a **negative quantity.**



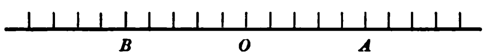
Note. If a quantity has no sign before it, the sign $+$ is understood. The sign $-$ is never omitted.

The idea of opposition can be very well illustrated by reference to a thermometer. The numbers above the zero mark are considered *positive* and those below the zero mark are *negative.* On a warm day in summer, the mercury may stand at 90° above zero; on a cold winter morning, it may stand at 10° below zero. The former is a positive quantity, and we write it $+90°$, or simply $90°$; the latter is a negative quantity, and we write it $-10°$.

Among other illustrations of the opposition of positive and negative quantities may be mentioned the following: if distance east of a given point is considered positive, distance west of the same point is negative; if north latitude is considered positive, south latitude is negative; if money received is considered positive, money paid out is negative. Thus we see that negative quantities are just as real as positive quantities.

In general, a quantity which when combined with a quantity under consideration will increase its value is considered a positive quantity, and a quantity which will similarly decrease its value is considered a negative quantity. It must be remembered, however, that in all cases it is a matter of conventionality as to which of two quantities opposite in nature shall be considered positive and which negative.

One of the best ways to obtain a good understanding of the relation between positive and negative quantities is found in the study of a horizontal line divided into equal spaces, each space representing a unit.



Let O denote the zero point. Then the spaces to the right of O are positive units, and those to the left of O are negative units. The value $+5$ will be represented by the point A, and the value -5 will be represented by the point B. Since the positive numbers increase in value as we count from left to right, progression in a positive direction will be toward the right hand, and progression in a negative direction will be toward the left hand.

In Arithmetic we recognize only the series of numbers on the right-hand side of the zero point, but in Algebra

we have the additional series on the left-hand side of the zero point. Accordingly, the operations of Algebra, which include both positive and negative numbers, are much more comprehensive than the corresponding operations of Arithmetic.

NOTE. Notwithstanding the two uses of the signs $+$ and $-$, there can be no confusion on this account, as the value of any expression is the same whichever interpretation we give to the signs.

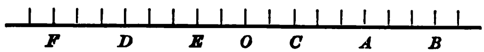
Every algebraic quantity consists of two parts, the sign and the absolute value. By the *sign* of a quantity we mean the sign $+$ or $-$ which is prefixed to it. The value independent of the sign $+$ or $-$ is called its *absolute value.*

If the signs of a quantity are both $+$ or both $-$, the quantities are said to have *like signs*; if one is $+$ and the other is $-$, the quantities are said to have *unlike signs.* When we *change the signs* of an expression, every $+$ changes to $-$ and every $-$ changes to $+$.

CHAPTER VII.

ADDITION.

20. The process of addition of simple expressions can be studied to good advantage by making use of a horizontal line divided as in § 19.



(i) Find the sum of $+5$ and $+3$. The point A represents $+5$. $+3$ denotes a progression of 3 units to the right. Hence the sum of $+5$ and $+3$ is represented by the point B, and is $+8$.

(ii) Find the sum of $+5$ and -3. The point A represents $+5$. -3 denotes a progression of 3 units to the left. Hence the sum of $+5$ and -3 is represented by the point C, and is $+2$.

(iii) Find the sum of -5 and $+3$. The point D represents -5. $+3$ denotes a progression of 3 units to the right. Hence the sum of -5 and $+3$ is represented by the point E, and is -2.

(iv) Find the sum of -5 and -3. The point D represents -5. -3 denotes a progression of 3 units to the left. Hence the sum of -5 and -3 is represented by the point F, and is -8.

From an investigation of these different cases we learn the following:

To add two numbers having like signs, *find their numerical sum, and prefix the common sign.* To add two numbers having unlike signs, *find their numerical difference, and prefix the sign of the larger number.*

NOTE. The sum of two numbers having the same absolute value and unlike signs is zero; thus, the sum of $+5$ and -5 is 0.

In each case the sum given is the *algebraic sum*; the sum of the absolute values is the *arithmetical sum.*

The units of division on the horizontal line may be of any length whatever. Let us suppose each space to denote a yard. Then, designating a yard by the letter y, the sum of $+5y$ and $+3y$ is $+8y$; the sum of $+5y$ and $-3y$ is $+2y$; the sum of $-5y$ and $+3y$ is $-2y$; and the sum of $-5y$ and $-3y$ is $-8y$. In place of y we may have any other letter or group of letters with no change in the numerical coefficients. Hence, to find the sum of two similar monomials, *add the coefficients, and annex to their sum the common letter or letters.*

To find the sum of more than two similar monomials, *find the sum of two of them, to their result add a third, and so on;* or *find the sum of the positive terms and the sum of the negative terms, and then find the algebraic sum of these two results.*

To add monomials which are not all similar, *add each group of similar terms, and unite these results by the proper signs.*

I. Find the sum of $4a^2b$, $-7a^2b$, $2a^2b$, $-a^2b$, and $5a^2b$.

Taking the coefficients in order, we think as follows: 4, -3, -1, -2, 3. Then we put down as the answer $3a^2b$.

If it seems preferable, we can take the positive terms and the negative terms separately. The sum of the positive terms is $11a^2b$, and the sum of the negative terms is $-8a^2b$. The sum of $11a^2b$ and $-8a^2b$ is $3a^2b$.

NOTE. Remember that the coefficient 1 is always understood when there is no coefficient expressed.

II. Find the sum of $9a^3$, $-ax^2$, $-4x^3$, $-2a^3$, $-3ax^2$, $9a^2x$, $-6a^3$, and $4x^3$.

The sum of $9a^3$, $-2a^3$, and $-6a^3$ is a^3; the sum of $-ax^2$ and $-3ax^2$ is $-4ax^2$; the sum of $-4x^3$ and $4x^3$ is 0. Uniting these results and the monomial $9a^2x$, the sum is $a^3 + 9a^2x - 4ax^2$.

NOTE 1. When the sum of any number of terms is 0, they may be dropped from consideration, as they do not affect the result.

NOTE 2. In writing polynomials containing different powers of the same letter, it is well to form the habit of writing the terms according to the descending or ascending powers of that letter.

EXAMPLES.

Find the sum of

1. a, $2a$, and $3a$.
2. $-2b$, $-5b$, and $-9b$.
3. $4xy$, $-3xy$, and $6xy$.
4. $3c^2d$, $-5c^2d$, and $-c^2d$.
5. $-8mn^3$, $5mn^3$, and $2mn^3$.
6. $10a^4b$, $-5a^4b$, and $-4a^4b$.
7. $-6abc$, abc, and $-8abc$.
8. $-5xy^3$, $-3xy^3$, and $6xy^3$.
9. $6xyz^2$, $-7xyz^2$, $-5xyz^2$, and $10xyz^2$.
10. $-4a^4b$, $8a^4b$, $3a^4b$, and $-7a^4b$.
11. $6cz$, $5cz$, $-8cz$, cz, and $-5cz$.
12. $-8m^3n$, $4m^3n$, $6m^3n$, $-5m^3n$, and $2m^3n$.
13. $2a$, $-4b$, $5c$, $-3b$, $6a$, and c.
14. $7a$, $-8x$, $-y$, $-7x$, $3a$, and $2x$.
15. $10x$, $3y$, z, $-4x$, $5y$, $2x$, and $-8z$.
16. $3ax$, $8bx$, 7, $-3bx$, -4, $3bx$, and 5.
17. a^3, $-5a^2b$, $3a^3$, $-6ab^2$, $-7b^3$, $-3ab^2$, and $-8a^2b$.

18. $3x^4$, $-6x^3$, $4x$, x^2, $-6x$, $-5x^4$, and $6x^3$.

19. $12c^2d$, $-6c^2$, $8d$, $-5c^2d$, $4d$, $-c^2$, and $-9c^2d$.

20. $9a^2bc$, $-3ab^2c$, $5abc^2$, $4abc$, $-4a^2bc$, $-5abc^2$, and $3ab^2c$.

21. $2x^2$, $3xy$, $-5y^2$, $-6x^2$, $-9xy$, $6y^2$, $-x^2$, and $-2xy$.

22. $2m^4$, $8m^2n^2$, $-4n^4$, $-3m^3n$, $5mn^3$, $-2m^2n^2$, $-mn^3$, $2m^3n$, and $-6m^2n^2$.

Simplify the following expressions by combining like terms:

23. $4a-7a+3a-5a+8a$.

24. $2m-5m-3m+m-9m$.

25. $8x^3-5x^3+7x^3-3x^3+3x^3-6x^3$.

26. $-14bx+12bx-3bx+bx-7bx$.

27. $5a^4c-9a^4c-11a^4c+8a^4c-2a^4c$.

28. $5xy^2-9xy^2+3xy^2-xy^2+4xy^2-2xy^2$.

29. $8ab-10cd+2ab+cd-6ab$.

30. $3ax-8bx+cx+8bx-ax+2bx$.

31. $x^3-3x^2y+3xy^2-y^3+4x^3-4y^3-6x^2y$.

32. $-3x^2+2xy-4z^2+3xz+x^2-6xy+z^2$.

33. $3ax^2-3by^2+6cz^2-4by^2-3cz^2+ax^2+5by^2$.

34. $4abc-7acd+10abd-4acd-8abd-abc+2abd$.

35. $4a^4-3a^2b^2+5b^4-3a^2b^2-3a^4-5b^4+6a^2b^2$.

36. $-m^4-3mn+4m^2p+8m^4-7mn+3m^4+11mn$.

ADDITION OF POLYNOMIALS.

21. Polynomials may be considered as made up of monomials. For example, $3a + 4b - 7c$ is the sum of the monomials $3a$, $4b$, and $-7c$. Hence the process of adding polynomials is the same as adding monomials which are not all similar.

I. Find the sum of $5x^4 - 7x^3 + 6x^2 + 8x$, $5x^3 - 7x + x^4$, $3x^2 - 3x - 5 - 4x^3$, and $6x - 9x^2 + 5x^3 - 3x^4$.

$$\begin{array}{rrrrr}
5x^4 & -7x^3 & +6x^2 & +8x & \\
x^4 & +5x^3 & & -7x & \\
 & -4x^3 & +3x^2 & -3x & -5 \\
-3x^4 & +5x^3 & -9x^2 & +6x & \\
\hline
3x^4 & -x^3 & & +4x & -5
\end{array}$$

For convenience it is customary to write the expressions under each other arranging similar terms in the same column. Adding each column separately, we find the entire sum to be $3x^4 - x^3 + 4x - 5$. Since the sum of the terms in the third column is 0, there is no x^2 term in the answer.

There is no carrying as in Arithmetic; hence in Algebra it is not necessary to begin at the right. In general it is more convenient to begin at the left. In fact, the usual way in Algebra is to begin at the left.

EXAMPLES.

Find the sum of

1. $3a + 8b$ and $3a - 8b$.

2. $2m - 1$ and $m^2 - 2m + 1$.

3. $3c^2 - 6c + 3$ and $c^2 + 7c - 7$.

4. $2a + 7b$, $-3a - 5b$, and $8a - 10b$.

5. $xy - xz + yz$, $xz - yz + xy$, and $yz - xy + xz$.

6. $4x - 3y + 2z$, $y - 2z - 5x$, and $x + 2y$.

7. $x^2 - 3x - 4$, $3x^2 - 5x + 4$, and $5x^2 + 8x - 6$.

8. $5a - 6b + 7c$, $3a + 7b - c$, and $a - 12b + 3c$.

9. $2p - 3q + 4r$, $4p + 6q - r$, and $p - 2q - 6r$.

10. $3a - 2b + c$, $4a + 3b - 4d$, and $-b + 3c - 2d$.

11. $2m - 3n - 4p$, $-5m - 6n + 8p - q$,
and $-4m + 9n - p$.

12. $x^3 - x^2 - x + 1$, $2x^2 - 4x - 5$, and $-4x^3 - 6x + 7$.

13. $a^2 + b^2 - 2ab$, $2a^2 + 5c^2 - 4ac$, and $2b^2 - 5c^2 + 8bc$.

14. $6x - 3y$, $5y - x$, $-8x - 8y$, and $9x + 2y$.

15. $a - b - x + y$, $2a - 3b + 4x - 5y$, $3a - x + 7y$,
and $3x - 3y$.

16. $7a - 5b - 3x + 7y$, $-6a + 8b - 4x - 3y$,
$3a - b + 2x - y$, and $a - 3b + 8x + 2y$.

17. $a^3 - 5a^2 + 6a - 3$, $3a^3 + a^2 - 7a - 2$,
$-5a^3 - 4a^2 + a$, and $6a^3 - 8a^2 + 9a + 7$.

18. $4x^3 - 3x^2y - 2xy^2 + y^3$, $7x^3 - 8x^2y - 3y^3$,
$4x^2y - 5xy^2 + 8y^3$, and $4x^3 + 6y^3$.

19. $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$, $6a^3b - 8a^2b^2 + 6ab^3$,
$2a^4 + 5a^2b^2 - 7ab^3$, and $8a^3b + a^2b^2$.

20. $8c^3 - 7c^2 - 8c + 11$, $7c^3 + 8c^2 - 5c + 3$,
$-9c^3 - 3c^2 + 7c$, and $2c^2 + 6c - 10$.

21. $y^3 - 4y^2 + 7y + 3$, $6y^3 + 6y^2 - 8y - 7$,
$3y - 3y^3 - 3y^2 + 12$, and $5y^2 + 3y - y^3 - 9$.

22. $15x^4 - 6x^2 - 5$, $x^3 - x + 3$, $x + 3x^2 - x^3 - 8x^4$,
and $3x^2 - 7x + 2$.

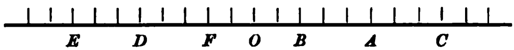
23. $4a - 3b + 6c - 4d$, $-7a + 2b - 3c + 8d$,
$a - 8b + 7c - 11d$, and $2a + 9b - 10c + 7d$.

24. $4a^3 - 3ab^2 + b^3$, $2ab^2 - 6a^2b - 3a^3$,
$4b^3 - 3a^3 + 8a^2b$, and $2a^2b - b^3 - a^3$.

CHAPTER VIII.

SUBTRACTION.

22. The process of subtraction of simple expressions can be studied in the same way that we studied addition.



Since subtraction is opposite to addition in its nature, subtracting a positive quantity indicates a progression to the left, and subtracting a negative quantity indicates a progression to the right.

(i) From $+5$ subtract $+3$. The point A represents $+5$. Subtracting $+3$ indicates a progression of 3 units to the left. Hence the remainder is represented by the point B, and is $+2$.

(ii) From $+5$ subtract -3. The point A represents $+5$. Subtracting -3 indicates a progression of 3 units to the right. Hence the remainder is represented by the point C, and is $+8$.

(iii) From -5 subtract $+3$. The point D represents -5. Subtracting $+3$ indicates a progression of 3 units to the left. Hence the remainder is represented by the point E, and is -8.

(iv) From -5 subtract -2. The point D represents -5. Subtracting -3 indicates a progression of 3 units to the right. Hence the remainder is represented by the point F, and is -2.

An investigation of these results shows that subtracting a positive number gives the same result as adding an equal negative number, and subtracting a negative number gives the same result as adding an equal positive number. Hence, to subtract one number from another, *change mentally the sign of the subtrahend, and then proceed as in addition.*

As in addition, the principle deduced for numbers can be shown to be true for monomials in general.

I. Subtract $9a$ from $7a$.

Changing the sign of $9a$, we have $-9a$. The sum of $7a$ and $-9a$ is $-2a$. Hence the remainder is $-2a$.

II. Subtract $-4a^3b$ from $7a^3b$.

Changing the sign of $-4a^3b$, we have $4a^3b$. The sum of $7a^3b$ and $4a^3b$ is $11a^3b$. Hence the remainder is $11a^3b$.

EXAMPLES.

Subtract

1. $4a$ from $10a$.
2. $8b$ from $2b$.
3. $-7c$ from $-3c$.
4. $-2d$ from $-9d$.
5. $4cd^3$ from $-5cd^3$.
6. $-m^2n$ from $6m^2n$.
7. $8k^3$ from k^3.
8. $-11xy$ from $4xy$.
9. $-6m^2n^2$ from $-10m^2n^2$.
10. $8c^3z$ from $-c^3z$.
11. $-12pq^2$ from $-3pq^2$.
12. $-x^2yz$ from $6x^2yz$.
13. $4b^3xy$ from $12b^3xy$.
14. $-18a^5c^3$ from $-17a^5c^3$.
15. $8nv^2$ from $-12nv^2$.
16. $-7bcd^3$ from $-15bcd^3$.
17. $13a^2x^4$ from $4a^2x^4$.
18. $-18x^3y^4$ from x^3y^4.
19. $-12a^2b^2c$ from $-3a^2b^2c$.
20. $2pqx^2$ from $-9pqx^2$.

SUBTRACTION OF POLYNOMIALS.

23. Since a polynomial may be regarded as the sum of several monomials, the process of subtracting a polynomial can also be made a process of addition.

I. From $4a - 5b + 6c - 7d - 8e + 9f$ subtract $a + 2b + 7c - 9d - 8e + 2g$.

$$\begin{array}{rrrrrrr} 4a & -5b & +6c & -7d & -8e & +9f & \\ a & +2b & +7c & -9d & -8e & & +2g \\ \hline 3a & -7b & -c & +2d & & +9f & -2g \end{array}$$

Write the subtrahend under the minuend, keeping similar terms in the same column. Conceive the subtrahend to have its signs changed so as to read $-a-2b-7c+9d+8e-2g$. Then proceeding as in addition, the result is $3a-7b-c+2d+9f-2g$.

EXAMPLES.

1. From $8a - 4b + c$ subtract $3a - 6b - 5c$.
2. From $7a - 3b + 6c$ subtract $7a + 3b - 6c$.
3. From $m + 2n - 3p$ subtract $4m + 5n + 6p$.
4. From $5p - 4q - 3r$ subtract $5q - 3p + r$.
5. From $x - y - z$ subtract $6x - 7y + 8z$.
6. From $3x^2 - 4x + 7$ subtract $3 - 2x$.
7. From $a^3 + b^3$ subtract $3a^3 - 4a^2b - 4ab^2$.
8. From $x^4 + x^2y^2 + y^4$ subtract $3x^3y + x^2y^2 + 3xy^3$.
9. From $a^2bc - ab^2c - abc^2$
   subtract $4ab^2c - 4abc^2 + 3a^2bc$.
10. From $a^3 + 3a^2b + 3ab^2 + b^3$
    subtract $a^3 - 3a^2b + 3ab^2 - b^3$.
11. From $-2x^3 + 3x^2 - 4x - 5$
    subtract $4x^3 + 5x^2 - 7x - 2$.
12. From $x^2 - 2xy + y^2 - z^2$
    subtract $x^2 - y^2 + 2yz - z^2$.

13. From $a^4 - 2a^2 + 3a - 1$
subtract $1 - 7a + a^2 - 3a^3$.

14. From $a^2 + 4b^2 - 3c^2$
subtract $3a^2 - 2b^2 - 6c^2 + d^2$.

15. From $-2x^3 + 6x^2y - 3xy^2 - y^3$
subtract $7x^3 + 6x^2y + 3xy^2 + 4y^3$.

16. From $6m^2 + 5n^2 + 4p^2 + 3q^2$
subtract $3m^2 + 4n^2 - 5p^2 - 6q^2$.

17. From $8x^4 - 6x^3 + 4x^2 - 2x + 1$
subtract $x^4 + 4x^3 + 7x^2 - 9x - 8$.

18. From the sum of $5a - 3b + 4c$ and $8a - 15c$
subtract $12a - 3b - 5c$.

19. From $2a^2 - 8a - 5$ subtract the sum of
$6a^2 - 2a - 8$ and $3 - 7a^2$.

20. From the sum of $5x^3 - 3x$ and $x^2 + 2x$ subtract
the sum of $3x^3 - 5x + 4$ and $6 - 3x - x^2$.

21. Subtract $a^2 - 8a + 4$ from a^3, and add the
remainder to $4a^2 + 9a - 7$.

22. What must be added to $7x - 3y + 4z$ to make
$2x - 5y - z$?

23. To what must $3x^2 - 5x + 1$ be added to make
$x^3 + 3x^2 - 7x$?

24. What must be subtracted from $3a - 2b + 5c + d$ to
leave a remainder of $-4a - 3b + 7c - 5d$?

25. From what must $-a + 2b + 3c - 4d$ be subtracted
to leave a remainder of $-3a - 8b - 10c + 5d$?

26. From what must the sum of $a - 2b + c$ and
$3a + 4b - 6c$ be subtracted to leave a remainder of $2a - b$?

PARENTHESES.

24. We have learned in § 4 that *when parentheses preceded by the sign + are removed, the signs of the terms within the parentheses remain unchanged; when parentheses preceded by the sign − are removed, the sign of every term within the parentheses is changed.*

Sometimes one sign of aggregation is used within another; in such cases it is customary to use different forms to avoid confusion. In removing the signs of aggregation from an expression of this kind, it is best to remove them, one at a time, *beginning with the innermost.*

I. Simplify $5a - 6b - (3a - 5c) + (b - 5c)$.

$$\begin{aligned} &5a - 6b - (3a - 5c) + (b - 5c) \\ &= 5a - 6b - 3a + 5c + b - 5c = 2a - 5b. \end{aligned}$$

II. Simplify $9x - [5x - 3y - (-y - \overline{3x - 4y})]$

$$\begin{aligned} &9x - [5x - 3y - (-y - \overline{3x - 4y})] \\ &= 9x - [5x - 3y - (-y - 3x + 4y)] \\ &= 9x - [5x - 3y + y + 3x - 4y] \\ &= 9x - 5x + 3y - y - 3x + 4y = x + 6y. \end{aligned}$$

NOTE. The sign − which precedes the term $3x$ under the vinculum is the sign of the vinculum, not of $3x$; since $3x$ has no sign expressed, the sign + is understood.

EXAMPLES.

Simplify

1. $x^2 + 2x + 1 - (x^2 - 2x + 1)$.
2. $a - (a - b) - (a - b + c)$.
3. $c^2 - d^2 - (c^2 + cd + d^2) - (c^2 - cd + d^2)$.
4. $a + b - c - (a - b) - (b - c)$.
5. $2a - (5a - b) - (6a - 3b) + (a - 4b)$.

6. $6a^2 + \{3b^2 - 2c^2\} - [a^2 + 8b^2] - (3c^2 - 5b^2)$.

7. $8a - (3b - a) + [-a + b - 3c] - \{7b - 3c\}$.

8. $x^3 - (7x^2 + 6x) - (3x - 4x^3) - (6 - 5x)$.

9. $x - (3x - 2) - (4y - 3) - (-6z + 5)$.

10. $(x + y + z) - (x + y - z) + (x - y - z)$.

11. $m - \{2p - (3n + 2p) - m\}$.

12. $7x - [4 - (3x + 5) - 5x]$.

13. $7a - \{b + [8c - (3b - c)]\}$.

14. $3b - (7b - \overline{6b + 5b})$.

15. $8m - [4m - \overline{3m - 7m}]$.

16. $c + d - (c - d) - [c - d - (c + d) - c]$.

17. $y + [2y - \{3y + (4y - 5y)\}]$.

18. $[2xy - (x^2 + y^2)] - [x^2 - (2xy - y^2)]$.

19. $(a^2 - \{a^2 + b^2 - c^2\}) - (a^2 + \{a^2 - b^2 - c^2\})$.

20. $9a - [8a - \{7a - (6a - 5a)\}]$.

21. $-6a - [4b - \{3a - (5a - b)\}]$.

22. $\{a - [b + (c + 2x) - (a - b)]\}$.

23. $a - [2b + \{3c - 3a - (a + b)\} + 2a - (b + 3c)]$.

24. $3a - [b - a - (2a + b - \overline{a - b})]$.

25. $a - [b - c - (a - b) - 5c] - (2a - \overline{5b - c})$.

CHAPTER IX.

MULTIPLICATION.

25. Multiplication is indicated by the sign $\times$ or by a dot (.); thus the product of a and b may be written $a \times b$ or $a \cdot b$. A more usual method, however, is to omit the sign of multiplication and write the factors one after the other; as ab. When several factors are alike we use an exponent; thus, we write a^3 instead of aaa.

In order to multiply monomials, it is merely necessary to write one after the other. In many cases a better form for the product may be obtained by changing the order of the factors. Especially is this true of numbers, which should be brought together at the beginning and multiplied as in Arithmetic. For example, the product of $7ab$ and $3cd$ is $21abcd$.

When the same letter occurs more than once as a factor, we must make use of the *index law.*

$$a^2 = aa, \text{ and } a^5 = aaaaa.$$

Then $a^2 \times a^5 = aa \times aaaaa = aaaaaaa = a^7$.

The same principle is true, whatever the exponents; hence *the exponent of a letter in the product is equal to the sum of the exponents in the factors.*

NOTE. For the numerical part of multiplication, *multiply coefficients and add exponents.*

26. From an arithmetical standpoint, multiplication is a short process of addition, where all the numbers to be added are alike. The product is obtained by taking a number (the *multiplicand*) as many times as there are

units in another number (the *multiplier*). When we extend this idea so as to embrace negative quantities, the sign of the multiplier shows whether the numbers denoted by the multiplicand are to be added or subtracted, and the absolute value of the multiplier shows the number of times they are to be taken.

(i) Multiply $+5$ by $+3$. This is the same as
$+5+5+5$, or $+15$.

(ii) Multiply -5 by $+3$. This is the same as
$-5-5-5$, or -15.

(iii) Multiply $+5$ by -3. This is the same as
$-(+5)-(+5)-(+5)$, or -15.

(iv) Multiply -5 by -3. This is the same as
$-(-5)-(-5)-(-5)$, or $+15$.

Letting a and b represent any two numbers, a similar investigation gives

$$(+a)\times(+b)=+ab.$$
$$(-a)\times(+b)=-ab.$$
$$(+a)\times(-b)=-ab.$$
$$(-a)\times(-b)=+ab.$$

These results enable us to state the *law of signs* in multiplication as follows: *like signs give* $+$; *unlike signs give* $-$.

MULTIPLICATION OF MONOMIALS.

27. I. Multiply $-6a^3b^2$ by $3ab^5$.

$-6a^3b^2\times 3ab^5=-18a^4b^7$. The law of signs gives $-$. The product of the coefficients is 18. The index law gives a^4b^7 for the literal part of the product.

II. Find the product of $-2ab^3$, $4a^2c^3$, and $-5bc^2d$.

$(-2ab^3) \times 4a^2c^3 \times (-5bc^2d) = 40a^3b^4c^5d$. The product of any number of monomials can be written down at once. If the number of negative factors is even, the sign of the product is $+$; if the number of negative factors is odd, the sign of the product is $-$.

Note. In writing the product of monomials, first write the sign; then write the product of the numerical coefficients; finally, write the letters in the order in which they occur in the alphabet, each with the proper exponent.

EXAMPLES.

Find the product of

1. $8a$ and 3.
2. $4ab$ and $-7c$.
3. $9c^3$ and c^2.
4. $-6xy^2$ and $5x^3y^2$.
5. $10x^3y^2z$ and $5x^2y^2z^3$.
6. $-15mn$ and $-3m^2n^2$.
7. $-8a^2b^2c^2$ and $3a^7d$.
8. $-12x^4y^5$ and $-7xy^3z^5$.
9. $9u^2v^2$ and $-8uv$.
10. $-20b^2c^2m$ and $4bmn^3$.
11. $2a$, $3b$, and $4c$.
12. a^2, $4a^3$, and $-5a^6$.
13. mn, $2m^2n^2$, and $3m^3n^3$.
14. $-2xy^2$, $-3yz^2$, and $-x^2z$.
15. am^2x^3, $-3a^4m^2x^6$, and $-8a^2x^2$.
16. $8m^3n$, $-3mn^2p$, and $-5p^2$.
17. $-6c^3d^2$, $-4cd^3$, and $2bc^2d^2$.
18. $2a^2b^3c^5$, $3b^4c^6$, and $-5a^2d$.
19. $-5a^2b^3$, $8b^2c^5$, and $3a^3c^4$.
20. $2a^2$, $-3a^3$, $-4a^4$, and $-5a^5$.
21. $-x^2yz$, $-xy^2z$, $-xyz^2$, and $-3x^2y^2z^2$.
22. $3m^4n$, $-7mp^2q$, $2n^4p^3$, and $-npq^5$.

MULTIPLICATION OF POLYNOMIALS BY MONOMIALS.

28. In § 4 we learned that

$$a(b - c) = ab - ac.$$

Considering $b - c$ as the multiplicand and a as the multiplier, we see that each term of the polynomial is multiplied by the monomial. Hence, to multiply a polynomial by a monomial, *multiply each term of the polynomial by the monomial, and connect the results thus obtained by the proper signs.*

I. Multiply $6x^3 - 4xy + y^2z^4$ by $5x^2y$.

$$\begin{array}{l} 6x^3 - 4xy + y^2z^4 \\ \underline{5x^2y \qquad\qquad\qquad\quad} \\ 30x^5y - 20x^3y^2 + 5x^2y^3z^4 \end{array}$$

First, multiply $6x^3$ by $5x^2y$, and work toward the right hand, multiplying each term of the multiplicand by $5x^2y$.

EXAMPLES.

Multiply

1. $a^2 - 3ab$ by $3ab$.
2. $c^2 - 5cd$ by $-c^2$.
3. $m^2 - 4n^2$ by $4n$.
4. $p^2 - 6q^2$ by $-p^3$.
5. $-x^2 - x^2y^2$ by $-x^2$.
6. $-3a^2b^2 + b^4$ by $-abc$.
7. $a^2 + 2a + 5$ by 3.
8. $3p^2 - 8pq - 4q^2$ by -6.
9. $a^3 - 4a^2 - 7$ by $4a$.
10. $-3x^2 - 5x + 7$ by $-5x^2$.
11. $x^2 - y^2 - z^2$ by xyz.
12. $2n^5 - 5n^3 - 8n$ by $-4n^2$.
13. $-2x^4 + 3x^3 - 4x + 5$ by $6x^2y$.
14. $x^2 - y^2 + 2yz - z^2$ by $2yz$.
15. $x^3 - 3x^2y + 3xy^2 - y^3$ by $-3x^2y$.
16. $-3a^3 + 2a^2 - a + 10$ by $5a^4$.

MULTIPLICATION OF POLYNOMIALS BY POLYNOMIALS.

29. When the multiplier is a polynomial, the process of multiplying is similar to that used in Arithmetic when the multiplier contains more than one figure.

If we are to multiply $a+b+c$ by $x+y+z$, we can let M stand for $x+y+z$.

$$\begin{aligned}\text{Then}\quad &(a+b+c)\times(x+y+z)\\ &=(a+b+c)\,M\\ &=aM+bM+cM\\ &=a(x+y+z)+b(x+y+z)+c(x+y+z)\\ &=ax+ay+az+bx+by+bz+cx+cy+cz.\end{aligned}$$

Thus the multiplicand is multiplied by each term of the multiplier, and the sum of the partial products thus obtained is the entire product.

I. Multiply $4a-5b$ by $3a-7b$.

$$\begin{array}{l}4a-5b\\ 3a-7b\\ \hline 12a^2-15ab\\ \qquad\;\; -28ab+35b^2\\ \hline 12a^2-43ab+35b^2\end{array}$$

Multiplying the multiplicand by $3a$, the first partial product is $12a^2-15ab$. Multiplying by $-7b$, the second partial product is $-28ab+35b^2$. Place similar terms in the same column, and add the partial products. Thus the entire product is $12a^2-43ab+35b^2$.

II. Multiply $6x+2x^3-3x^2+x^4$ by $3x^2-2-x$.

$$\begin{array}{rrrrrr}
 & x^4 & +2x^3 & -3x^2 & +6x & \\
 & 3x^2 & -x & -2 & & \\ \hline
3x^6 & +6x^5 & -9x^4 & +18x^3 & & \\
 & -x^5 & -2x^4 & +3x^3 & -6x^2 & \\
 & & -2x^4 & -4x^3 & +6x^2 & -12x \\ \hline
3x^6 & +5x^5 & -13x^4 & +17x^3 & & -12x
\end{array}$$

By arranging both multiplicand and multiplier according to the descending powers of x, similar terms of the partial products readily fall in the same column. The same would be true if the polynomials were arranged according to the ascending powers of x.

EXAMPLES.

Multiply

1. $a + b$ by $a + b$.
2. $a - b$ by $a - b$.
3. $a + b$ by $a - b$.
4. $x + 9$ by $x + 4$.
5. $x + 9$ by $x - 4$.
6. $x - 9$ by $x + 4$.
7. $x - 9$ by $x - 4$.
8. $x + y$ by $x + z$.
9. $2a + 3b$ by $-2a - 3b$.
10. $3c + 5d$ by $-3c + 5d$.
11. $5 + 7m$ by $5 - 3m$.
12. $8a - 1$ by $-3a - 2$.
13. $m^2 + 3n$ by $m^2 - 5n$.
14. $3x^2y^2 - 8$ by $4x^2y^2 - 1$.
15. $a^2 - ab + b^2$ by $a + b$.
16. $a^2 + ab + b^2$ by $a - b$.
17. $a^2 - ab - b^2$ by $a - b$.
18. $p^2 + 2pq + q^2$ by $p - q$.
19. $p^2 - 2pq + q^2$ by $p - q$.
20. $-x^4 + x^3y - x^2y^2$ by $-x - y$.
21. $a + b - c$ by $a + c$.
22. $2x + y - z$ by $x - y$.
23. $3x^2 - 4x + 5$ by $2x - 3$.
24. $2a^2 - 3ab - 4b^2$ by $3a - 4b$.
25. $4a^2 + 6ab + 9b^2$ by $2a - 3b$.
26. $a^3 - 2a^2 + 4a - 8$ by $a + 2$.
27. $x^3 - 3x^2y + 3xy^2 - y^3$ by $x + 3y$.
28. $x^2 + xy + y^2$ by $x^2 - xy + y^2$.

29. $m^3 + 2m^2n + 2mn^2$ by $m^2 - 2mn + 2n^2$.

30. $6x - 1 + 8x^2$ by $6x - 1 - 8x^2$.

31. $x + y + z$ by $x + y - z$.

32. $x + 2y - 3z$ by $x - 2y + 3z$.

33. $c^4 - 3c^3 - c^2$ by $-c^2 + c + 1$.

34. $6a^3 - 2a^2b + 4ab^2$ by $2a^2b - 5ab^2 - 3b^3$.

35. $mn - 5m^2 + 8n^2$ by $mn - 2m^2 - 3n^2$.

36. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a^2 + 3ab + b^2$.

37. $2d^2 - 3d^3 + 8d + 2d^4$ by $3d - d^2 - 2$.

38. $k^5 + 5k^3 - 7k^2 - 3k^4$ by $3k^2 - 5k + k^3$.

39. $xy + x + y + 1$ by $xy - x - y + 1$.

40. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

SPECIAL CASES IN MULTIPLICATION.

20. There are certain products that occur so frequently that the student should learn to write these products by inspection.

I. Find the square of $a + b$.

$$\begin{array}{l} \quad a + b \\ \quad \underline{a + b \qquad\quad} \\ a^2 + ab \\ \underline{\qquad\quad ab + b^2} \\ a^2 + 2ab + b^2 \end{array}$$

That is, *the square of the sum of two quantities is equal to the square of the first quantity, plus twice the product of the two, plus the square of the second.*

II. Find the square of $a - b$.

$$\begin{array}{l} a - b \\ \underline{a - b} \\ a^2 - ab \\ \underline{\quad - ab + b^2} \\ a^2 - 2\,ab + b^2 \end{array}$$

That is, *the square of the difference of two quantities is equal to the square of the first quantity, minus twice the product of the two, plus the square of the second.*

III. Find the product of $a + b$ and $a - b$.

$$\begin{array}{l} a + b \\ \underline{a - b} \\ a^2 + ab \\ \underline{\quad - ab - b^2} \\ a^2 \qquad - b^2 \end{array}$$

That is, *the product of the sum and the difference of two quantities is equal to the difference of their squares.*

NOTE. To find the square of a monomial, *square the coefficient, and multiply the exponent of each letter by* 2. For example, the square of $9\,a^3b^5$ is $81\,a^6b^{10}$.

EXAMPLES.

Write by inspection the value of

1. $(x + y)^2$.
2. $(m - n)^2$.
3. $(c + 1)^2$.
4. $(z - 1)^2$.
5. $(m + 5)^2$.
6. $(c^2 - 4)^2$.
7. $(y^2 + z^2)^2$.
8. $(p^3 - q^3)^2$.
9. $(x^2 + 8)^2$.
10. $(x^2 - 2\,x)^2$.
11. $(2\,c + 3\,d)^2$.
12. $(8 - 5\,x)^2$.
13. $(3\,ab + 7\,c)^2$.
14. $(4\,x^2 - 5\,xy)^2$.

15. $(2a^2 + 11ab)^2$.

16. $(5mn - 9n^2)^2$.

17. $(c + d)(c - d)$.

18. $(x + 10)(x - 10)$.

19. $(x^2 + 1)(x^2 - 1)$.

20. $(mn + 7)(mn - 7)$.

21. $(2p + 3q)(2p - 3q)$.

22. $(2y^3 + 5)(2y^3 - 5)$.

23. $(5a^2 + 6b^2)(5a^2 - 6b^2)$.

24. $(8x^2 + 7x)(8x^2 - 7x)$.

31. The product of two binomials like $x + 7$ and $x + 3$ also needs attention.

$$\begin{array}{r} x + 7 \\ x + 3 \\ \hline x^2 + 7x \qquad \\ 3x + 21 \\ \hline x^2 + 10x + 21 \end{array} \qquad \begin{array}{r} x - 7 \\ x - 3 \\ \hline x^2 - 7x \qquad \\ - 3x + 21 \\ \hline x^2 - 10x + 21 \end{array}$$

$$\begin{array}{r} x + 7 \\ x - 3 \\ \hline x^2 + 7x \qquad \\ - 3x - 21 \\ \hline x^2 + 4x - 21 \end{array} \qquad \begin{array}{r} x - 7 \\ x + 3 \\ \hline x^2 - 7x \qquad \\ 3x - 21 \\ \hline x^2 - 4x - 21 \end{array}$$

From these results we deduce the following principle: *the product of two binomials having a common term is made up of three terms — 1st, the square of the common term; 2nd, the algebraic sum of the remaining terms multiplied by the common term; 3rd, the product of the remaining terms.*

EXAMPLES.

Write by inspection the value of

1. $(x + 2)(x + 3)$.

2. $(x - 7)(x + 1)$.

3. $(x - 7)(x - 8)$.

4. $(x + 6)(x - 3)$.

5. $(a - 10)(a + 3)$.

6. $(a + 6)(a + 4)$.

7. $(a-5)(a+4)$.

8. $(a-12)(a-2)$.

9. $(ab+6)(ab-5)$.

10. $(x^2+9)(x^2+7)$.

11. $(p^3+6)(p^3-12)$.

12. $(y^2z-3)(y^2z-8)$.

13. $(a-5b)(a+b)$.

14. $(x+3a)(x+11a)$.

15. $(x-5y)(x-4y)$.

16. $(c+7d)(c-d)$.

17. $(a^2+2x^2)(a^2+5x^2)$.

18. $(x^3+6a)(x^3-5a)$.

19. $(m^3-n^3)(m^3-4n^3)$.

20. $(x^2y+xy^2)(x^2y-2xy^2)$.

21. $(ab-2cd)(ab-5cd)$.

22. $(m^2-4pq)(m^2+8pq)$.

23. $(ax+7by)(ax-by)$.

24. $(abc+3d)(abc-9d)$.

32. When an algebraic expression contains any of the special cases in multiplication or the product of a polynomial by a monomial, it can generally be simplified without writing down each case of multiplication separately.

I. Simplify $(x+1)(x-7)-[(x-3)^2-(x+2)(x-2)]$.

$$\begin{aligned}
&(x+1)(x-7)-[(x-3)^2-(x+2)(x-2)]\\
=\ &x^2-6x-7-[x^2-6x+9-(x^2-4)]\\
=\ &x^2-6x-7-[x^2-6x+9-x^2+4]\\
=\ &x^2-6x-7-x^2+6x-9+x^2-4=x^2-20.
\end{aligned}$$

II. Simplify $12a-3[2b+\{a-4(5a-b)\}]$.

$$\begin{aligned}
&12a-3[2b+\{a-4(5a-b)\}]\\
=\ &12a-3[2b+\{a-20a+4b\}]\\
=\ &12a-3[2b+a-20a+4b]\\
=\ &12a-6b-3a+60a-12b=69a-18b.
\end{aligned}$$

EXAMPLES.

Simplify

1. $(x+7)(x-3)-(x-7)(x+3)$.
2. $(a+b)^2-(a-b)^2$.
3. $(a+b)^2(a-b)^2$.
4. $(x-y)^2+(y-z)^2+(z-x)^2$.
5. $(5a-6b)^2-6(a-3b)(a-2b)$.
6. $(2m-3n)^2-4(m-n)(m-2n)-n^2$.
7. $(a-2b)(a-3b)-(a-3b)(a-4b)-2ab$.
8. $x-(x-y)-[x+y-z-2(y-z)]$.
9. $a-2(b-c)-3[a-4(b+c)]$.
10. $3a-\{4b-2[3a+3(5-b)]\}$.
11. $16-2[3x-3\{4x-4(5x-6)\}]$.
12. $2(4a-6b)-7[b-4\{2-(3b-a)\}]$.
13. $5-3[6x-2\{-3x+5(6-\overline{2-x})\}]$.
14. $a[a-b-a(c-ab)]-[ac-a(ac+b)]$.
15. $x^2-[(x-y)^2-(x+y)(x-y)]$.
16. $(x+y)x-[(x-y)^2-(y-x)y]$.
17. $[a^2-(a-b)(a-2b)]-b[b-(a-b)]$.
18. $(x-2)(x-4)-3x(x-2)+2[(x+3)(x+2)-10]$.
19. $(x^2-y^2)z-(x-y)(x[y+z]-y[x-z])$.
20. $(m+n)x-(n-p)p-[(n-x)n-(n-p)(n+p)]-mx$.

CHAPTER X.

DIVISION.

33. Division is the process of finding one of two factors when the product and the other factor are known. Hence division is the inverse of multiplication.

As in Arithmetic, the product is called the *dividend*, the known factor is called the *divisor*, and the factor to be found is called the *quotient*.

The operation of division is indicated by the sign $\div$, or by writing the dividend above the divisor with a line between; thus, $a \div b$, or $\frac{a}{b}$ signifies that a is to be divided by b.

The index law and the law of signs in division can easily be deduced from the corresponding laws in multiplication.

Since $a^2 \times a^5 = a^7$, $a^7 \div a^2 = a^5$. The same principle is true whatever the exponents; hence *the exponent of a letter in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.* This is the *index law* in division.

NOTE. For the numerical part of division, *divide coefficients and subtract exponents.*

Since $(+a) \times (+b) = +ab$, $(+ab) \div (+a) = +b$.
Since $(-a) \times (+b) = -ab$, $(-ab) \div (-a) = +b$.
Since $(+a) \times (-b) = -ab$, $(-ab) \div (+a) = -b$.
Since $(-a) \times (-b) = +ab$, $(+ab) \div (-a) = -b$.

Hence in division, just as in multiplication, *like signs give* $+$; *unlike signs give* $-$. This is the *law of signs* in division.

DIVISION OF MONOMIALS.

34. I. Divide $abcd$ by cd.

$abcd \div cd = ab$ Since the dividend contains all the factors of both divisor and dividend, the quotient is derived by removing from the dividend all the factors that are found in the divisor.

II. Divide $-24\,a^5b^3$ by $3\,a^4b$.

$-24\,a^5b^3 \div 3\,a^4b = -8\,ab^2$ The law of signs gives $-$. 24 divided by 3 is 8. The index law gives ab^2 as the literal part of the quotient.

III. Divide $-12\,a^6b^4c^2d$ by $-2\,a^4b^4c^2$.

$-12\,a^6b^4c^2d \div (-2\,a^4b^4c^2) = 6\,a^2d$ The law of signs gives $+$. 12 divided by 2 is 6. The index law gives a^2. b^4c^2 drops out, as it is the same in both dividend and divisor, and d remains.

NOTE. $\frac{b^4}{b^4} = 1$. By the index law, however, $\frac{b^4}{b^4} = b^{4-4} = b^0$. Hence $b^0 = 1$. Any letter, which by the index law would have 0 for an exponent, does not appear in the quotient.

EXAMPLES.

Divide

1. $36\,a$ by 9.
2. $abcd$ by $-abc$.
3. $16\,ab$ by $4\,a$.
4. $-18\,ab$ by $9\,ab$.
5. $-32\,abc$ by $-4\,ab$.
6. a^4b^3c by a^2b^2.
7. a^3bc by $-a^3bc$.
8. $-12\,c^5$ by c^2.
9. $-8\,x^4y^2$ by $8\,x^2$.
10. $-24\,m^2n^2$ by $-6\,mn$.
11. $90\,p^4q$ by $15\,pq$.
12. $20\,x^6y^3$ by $-5\,x^2y^2$.
13. $-45\,c^3d^3x^3$ by $-3\,c^2d^2$.
14. $-12\,xyz^2$ by $-3\,xyz$.
15. $42\,m^2nx^2$ by $7\,mn$.
16. $-m^7n^5p^3$ by m^7n^4p.
17. $20\,x^3y^4z^2$ by $-4\,x^2y^2z$.
18. $-26p^2q^2r^2$ by $-2pq^2r^2$.
19. $72\,a^5b^3c^5$ by $-9\,ab^3c^5$.
20. $-75\,abc^2d^2$ by $15\,acd$.

DIVISION OF POLYNOMIALS BY MONOMIALS

35. Since [illegible]

[illegible]

The [illegible] by [illegible] a [illegible] the proper [illegible]

1. Divide [illegible] by [illegible].

[illegible] Each term of the dividend is divided by [illegible].

EXAMPLES.

Divide

1. $3x^5 - 4x^3$ by x^2.
2. $5m^2n - 10n^2$ by $-5n$.
3. $-a^4 + 4$ [illegible] by [illegible].
4. 12 [illegible] $+ 8$ [illegible] by 4 [illegible].
5. $-5x^4y^2z^2 - 3x^2yz$ by $-xyz$.
6. $21a^4b^3 - 7a^3b^2$ by $7a^3b^2$.
7. $12a^2b + 18a^4c^2 - 21c^5$ by 3.
8. $16m^2n - 12mp^4 - 20p^5$ by -4.
9. $x^2yz + xy^2z - xyz^2$ by $-xyz$.
10. $x^7 - 2x^6 + 3x^5$ by x^3.
11. $3c^2y^4 - 15c^4y^3 - 9c^4y$ by $3cy$.
12. $4y^6 - 6y^4 + 2y^3$ by $2y^3$.
13. $-26x^4 - 39x^3 + 52x^2$ by $-13x$.
14. $-21a^3m^4 + 35b^2m^5 - 42cm^6 + 14d^4m^8$ by $7m^4$.
15. $-5a^6b^4 - 10a^5b^5 + 25a^8b^6$ [illegible] $35a^{10}b^8$ by $5a^4b^4$.
16. $28a^9b^4c^4 + 20a^4b^5c^6 - 12a^6b^5c^5$ [illegible] $4a^6b^7c^4$ by $4a^4b^4c^4$.

DIVISION OF POLYNOMIALS BY POLYNOMIALS.

36. As the product of two polynomials is made up of several partial products, the dividend can be separated into several parts, each of which can be divided by the divisor. The operation can be understood more easily by studying a particular example.

I. Divide $3x^2 + 19xy + 28y^2$ by $x + 4y$.

$$\begin{array}{l} x+4y\,)\,3x^2+19xy+28y^2\,(\,3x+7y \\ \qquad\quad \underline{3x^2+12xy} \\ \qquad\qquad\quad\ 7xy+28y^2 \\ \qquad\qquad\quad\ \underline{7xy+28y^2} \end{array}$$

The dividend and divisor are arranged according to the descending powers of x. As the first term of a product is obtained by multiplying the first term of the multiplicand by the first term of the multiplier, the first term of the quotient can be found by dividing the first term of the dividend by the first term of the divisor. $3x^2 \div x = 3x$; hence the first term of the quotient is $3x$. Multiplying $x + 4y$ by $3x$, we have $3x^2 + 12xy$ as one of the parts into which the dividend can be separated. By subtraction, we have $7xy + 28y^2$, which must be the product of the divisor and the rest of the quotient. Then the second term of the quotient is $7xy \div x$, or $7y$. Multiplying $x + 4y$ by this term, we have $7xy + 28y^2$, and there is no remainder. Hence the entire quotient is $3x + 7y$.

From the above example we see that the process of dividing one polynomial by another is as follows:

(i) *Arrange both dividend and divisor according to the descending (or ascending) powers of some common letter.*

(ii) *Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.*

(iii) *Multiply the divisor by the first term of the quotient, and subtract the product from the dividend.*

(iv) *If there is a remainder, treat it as a new dividend and proceed as before, continuing the process until there is no remainder.*

II. Divide $a^4 - a^3b + ab^3 - b^4$ by $a^2 - b^2$.

$$\begin{array}{l}
a^2 - b^2)\,a^4 - a^3b + ab^3 - b^4\,(a^2 - ab + b^2 \\
\underline{a^4 \qquad - a^2b^2} \\
\quad - a^3b + a^2b^2 + ab^3 - b^4 \\
\quad \underline{- a^3b \qquad\quad + ab^3} \\
\qquad\quad a^2b^2 \qquad - b^4 \\
\qquad\quad \underline{a^2b^2 \qquad - b^4}
\end{array}$$

III. Divide $x^6 + 27$ by $x^2 + 3$.

$$\begin{array}{l}
x^2 + 3)\,x^6 + 27\,(x^4 - 3x^2 + 9 \\
\underline{x^6 + 3x^4} \\
\quad - 3x^4 + 27 \\
\quad \underline{- 3x^4 - 9x^2} \\
\qquad\quad 9x^2 + 27 \\
\qquad\quad \underline{9x^2 + 27}
\end{array}$$

IV. Divide $a^2 - b^2 - c^2 + 2bc$ by $a - b + c$.

$$\begin{array}{l}
a - b + c)\,a^2 - b^2 - c^2 + 2bc\,(a + b - c \\
\underline{a^2 - ab + ac} \\
\quad ab - ac - b^2 - c^2 + 2bc \\
\quad \underline{ab \qquad - b^2 \qquad + bc} \\
\qquad - ac \qquad - c^2 + bc \\
\qquad \underline{- ac \qquad - c^2 + bc}
\end{array}$$

A more compact arrangement of work may be obtained by writing the divisor at the right of the dividend with the quotient below the divisor.

V. Divide $6x^3 - 25x^2y + 27xy^2 - 5y^3$ by $2x - 5y$.

$$\begin{array}{l|l}
6x^3 - 25x^2y + 27xy^2 - 5y^3 & 2x - 5y \\ \cline{2-2}
\underline{6x^3 - 15x^2y} & 3x^2 - 5xy + y^2 \\
\quad - 10x^2y + 27xy^2 - 5y^3 & \\
\quad \underline{- 10x^2y + 25xy^2} & \\
\qquad\qquad 2xy^2 - 5y^3 & \\
\qquad\qquad \underline{2xy^2 - 5y^3} &
\end{array}$$

NOTE. Every remainder should be arranged according to the descending (or ascending) powers of the same letter as the dividend and divisor.

EXAMPLES.

Divide

1. $a^2 + 2ab + b^2$ by $a + b$.
2. $a^2 - 2ab + b^2$ by $a - b$.
3. $a^2 - b^2$ by $a + b$.
4. $x^2 + 8x + 15$ by $x + 3$.
5. $x^2 - 10x + 16$ by $x - 2$.
6. $x^2 + 3x - 18$ by $x + 6$.
7. $x^2 - 5x - 24$ by $x - 8$.
8. $12x^2 + 11xy - 56y^2$ by $4x - 7y$.
9. $a^3 + x^3$ by $a + x$.
10. $a^4 - x^4$ by $a - x$.
11. $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$.
12. $a^5 + 32b^5$ by $a + 2b$.
13. $a^6 - x^6$ by $a^2 + ax + x^2$.
14. $c^7 + c$ by $c^2 + 1$.
15. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.
16. $a^2 - 2ab + b^2 - c^2$ by $a - b + c$.
17. $a^2 - b^2 - c^2 - 2bc$ by $a + b + c$.
18. $4x^2 - 9y^2 + 6yz - z^2$ by $2x + 3y - z$.
19. $15m^3 - 10m^2n - 9mn^2 + 6n^3$ by $3m - 2n$.
20. $30k^3 - 43k + 11k^2 + 12$ by $5k - 4$.
21. $21a^2x^2 - 34abxy + 8b^2y^2$ by $3ax - 4by$.
22. $a^6 - 4a^4b^2 + 4a^2b^4 - b^6$ by $a^2 - b^2$.
23. $a^5 + a^3b^2 + ab^2 - a^4b - a^2b^3 - b^5$ by $a^3 - b^3$.
24. $x^2 + xy + 3xz - 2y^2 + 6yz$ by $x + 2y$.
25. $x^3 - 3x^2y + 3xy^2 - y^3$ by $x^2 - 2xy + y^2$.
26. $6a^4 + 9a^2 - 15a$ by $2a^2 + 2a + 5$.
27. $a^4 + 4b^4$ by $a^2 + 2ab + 2b^2$.

28. $c^3 + 3c^2d - 16cd^2 + 12d^3$ by $c^2 + 5cd - 6d^2$.

29. $d^4 - 31d^2 + 9$ by $d^2 + 5d - 3$.

30. $m^4 - 18m^2 + 81$ by $m^2 - 6m + 9$.

31. $a^4 + 2 + a + 2a^2$ by $a + 1 + a^2$.

32. $3h^4 + h + 2h^2 - h^3 + 1$ by $h^2 + 1 - h$.

33. $p^4 - 5p^2 + 14p - 12$ by $p^2 - 2p + 3$.

34. $10x^3 - 7x - 11x^2 + 6x^4 + 2$ by $4x + 2x^2 - 1$

35. $12x^2 + 3x^4 - 8x^3 - 16$ by $x^2 + 4 - 2x$.

36. $a^6 - 2a^3b^3 + b^6$ by $a^2 - 2ab + b^2$.

37. $c^5 + 5c^2 - 5c^3 - 1$ by $c^2 + 3c + 1$.

38. $1 - 3x^4 + 2x^6$ by $1 + 2x + x^2$.

39. $x^2y^2 - x^2 - y^2 + 1$ by $xy - x - y + 1$.

40. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

41. $x^5 + x^4y - x^3y^2 + x^2 - 2xy^2 + y^3$ by $x^3 + x - y$.

42. $2x^4 + xy - xz - 3y^2 - 4yz - z^2$ by $2x + 3y + z$.

SPECIAL CASES IN DIVISION.

37. The following results can easily be verified:

$$(a^2 - b^2) \div (a + b) = a - b.$$
$$(a^2 - b^2) \div (a - b) = a + b.$$
$$(a^3 + b^3) \div (a + b) = a^2 - ab + b^2.$$
$$(a^3 - b^3) \div (a - b) = a^2 + ab + b^2.$$

These examples illustrate the following principles:

(i) *If the difference of the squares of two quantities is divided by the sum of the quantities, the quotient is the difference of the quantities.*

(ii) *If the difference of the squares of two quantities is divided by the difference of the quantities, the quotient is the sum of the quantities.*

(iii) *If the sum of the cubes of two quantities is divided by the sum of the quantities, the quotient is the square of the first quantity, minus the product of the two, plus the square of the second.*

(iv) *If the difference of the cubes of two quantities is divided by the difference of the quantities, the quotient is the square of the first quantity, plus the product of the two, plus the square of the second.*

EXAMPLES.

Write by inspection the value of

1. $(x^2 - y^2) \div (x + y)$.
2. $(x^2 - 1) \div (x - 1)$.
3. $(4x^2 - 9y^2) \div (2x + 3y)$.
4. $(16a^2 - 1) \div (4a - 1)$.
5. $(c^2 - 36d^2) \div (c + 6d)$.
6. $(9a^2 - 49d^2) \div (3a - 7d)$.
7. $(1 - 81a^2b^2) \div (1 + 9ab)$.
8. $(25m^2 - 81n^2) \div (5m - 9n)$.
9. $(4y^6 - 25) \div (2y^3 + 5)$.
10. $(64x^6 - 25x^2) \div (8x^3 - 5x)$.
11. $(x^3 + y^3) \div (x + y)$.
12. $(x^3 - 1) \div (x - 1)$.
13. $(a^3 + 8b^3) \div (a + 2b)$.
14. $(27c^3 - d^3) \div (3c - d)$.
15. $(1 + 8x^3) \div (1 + 2x)$.
16. $(64m^3 - 1) \div (4m - 1)$.
17. $(125p^3 + 8) \div (5p + 2)$.
18. $(a^6 + b^6) \div (a^2 + b^2)$.
19. $(8c^6 + 27d^3) \div (2c^2 + 3d)$.
20. $(8x^9 - y^6) \div (2x^3 - y^2)$.

CHAPTER XI.

SIMULTANEOUS EQUATIONS.

38. If we have an equation containing two unknown quantities, we can find an unlimited number of pairs of values which satisfy the equation. For example, consider the equation $3x + y = 7$. Transposing the term $3x$, we have $y = 7 - 3x$. Then if $x = 1$, $y = 4$; if $x = 2$, $y = 1$; if $x = 3$, $y = -2$; and so on without limit.

Likewise, a second equation of the same kind, as $x + 4y = 6$, can be satisfied by an unlimited number of pairs of values.

If, however, we consider both the equations mentioned above, we find that $x = 2$ and $y = 1$ is the only pair of values which will satisfy both equations.

Two equations which express different relations between two unknown quantities are called **independent equations.** For example, $3x + y = 7$ and $x + 4y = 6$ are independent equations. $3x + y = 7$ and $6x + 2y - 4 = 10$ are not independent equations; by transposing -4 and dividing by 2, the latter equation becomes the same as the first.

Two or more independent equations which contain the same unknown quantities are called **simultaneous equations.** In order to solve such equations, there must be just as many equations as there are unknown quantities.

The process of deriving from two or more equations an equation containing one less unknown quantity than the given equations is called **elimination**, and the quantity which does not appear in the new equation is said to have been *eliminated.*

I. Solve $\begin{cases} 2x + 5y = 26. \\ 5x - 2y = 7. \end{cases}$

Multiplying the first equation by 2 and the second equation by 5,

$$4x + 10y = 52.$$
$$25x - 10y = 35.$$

By addition,
$$29x = 87.$$
$$x = 3.$$

Substituting this value of x in the first equation,

$$6 + 5y = 26.$$
$$5y = 20.$$
$$y = 4.$$

II. Solve $\begin{cases} 6x + 7y = 35. \\ 4x - 3y = 31. \end{cases}$

Multiplying the first equation by 2 and the second equation by 3,

$$12x + 14y = 70.$$
$$12x - 9y = 93.$$

By subtraction,
$$23y = -23.$$
$$y = -1.$$

Substituting this value of y in the first equation,

$$6x - 7 = 35.$$
$$6x = 42.$$
$$x = 7.$$

In I. the equations were added to eliminate y, because the signs of the y terms were unlike. In II. the equations were subtracted to eliminate x, because the signs of the x terms were alike.

NOTE. This method of elimination is known as *elimination by addition or subtraction.* There are several other methods of elimination, but this is the best one for simple equations.

It is always best to eliminate the letter which will require the smallest multipliers to make the coefficients equal.

In finding the value of the second unknown quantity, the value of the one found first can be substituted in either of the original equations.

If the equations contain fractions or parentheses, each equation should be reduced to its simplest form before eliminating.

EXAMPLES.

Solve the following equations:

1. $\begin{cases} x + y = 12. \\ x - y = 2. \end{cases}$

2. $\begin{cases} x + 3y = 19. \\ x - 2y = 4. \end{cases}$

3. $\begin{cases} 6x - y = 8. \\ 3x + y = 19. \end{cases}$

4. $\begin{cases} 3x + 2y = 23. \\ 4x - 3y = 8. \end{cases}$

5. $\begin{cases} 5x + 2y = 19. \\ 3x + 4y = 17. \end{cases}$

6. $\begin{cases} 2x - 3y = 14. \\ 4x - 5y = 20. \end{cases}$

7. $\begin{cases} 4x + 3y = 17. \\ 3x + 4y = 39. \end{cases}$

8. $\begin{cases} 2x + 3y = 34. \\ 7x - 9y = 80. \end{cases}$

9. $\begin{cases} 7x + 3y = 22. \\ 9x + 4y = 29. \end{cases}$

10. $\begin{cases} 5x + 4y = 17. \\ 4x + 3y = 12. \end{cases}$

11. $\begin{cases} 3x - 7y = 0. \\ 5x + 3y = 44. \end{cases}$

12. $\begin{cases} 6x - 8y = 24. \\ 5x - 7y = 18. \end{cases}$

13. $\begin{cases} 12x + 5y = 40. \\ 8x - 3y = 52. \end{cases}$

14. $\begin{cases} 7x + 3y = 20. \\ 5x - 6y = 55. \end{cases}$

15. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y = 10. \\ \frac{1}{4}x - \frac{1}{2}y = 1. \end{cases}$

16. $\begin{cases} \frac{2}{3}x - \frac{1}{5}y = 6. \\ \frac{1}{4}x + 2y = 23. \end{cases}$

17. $\begin{cases} \dfrac{x}{4} + 4y = 35. \\ 4x + \dfrac{y}{4} = 50. \end{cases}$

18. $\begin{cases} \dfrac{3x}{4} + \dfrac{4x}{3} = 34. \\ \dfrac{4x}{3} - \dfrac{3x}{4} = 23. \end{cases}$

19. $\begin{cases} \dfrac{x - 2y}{2} + x = 7. \\ \dfrac{x + y}{4} + y = 4. \end{cases}$

20. $\begin{cases} \dfrac{2x - 4y}{7} = 4. \\ \dfrac{5x + 6y}{2} = 11. \end{cases}$

21. $\begin{cases} \dfrac{1}{x} - \dfrac{3}{y} = 0. \\ \dfrac{x+y}{4} = \dfrac{2y}{3} - 6. \end{cases}$

25. $\begin{cases} \dfrac{x+2}{3} - 8y = 31. \\ 10x - \dfrac{y-5}{4} = 192. \end{cases}$

22. $\begin{cases} \dfrac{4x}{7y} = 1\frac{1}{3}. \\ \dfrac{x+5}{4} = \dfrac{5y}{6} + \dfrac{1}{2}. \end{cases}$

26. $\begin{cases} \dfrac{x+y}{3} - \dfrac{x-y}{3} = 6. \\ \dfrac{x-y}{2} + \dfrac{x+y}{8} = 6. \end{cases}$

23. $\begin{cases} \dfrac{x-2}{4} - \dfrac{y+3}{5} = 0. \\ \dfrac{2x-5}{5} - \dfrac{13-y}{2} = 0. \end{cases}$

27. $\begin{cases} \dfrac{x-8y}{4} + y = \dfrac{x}{5}. \\ \dfrac{x}{5} - \dfrac{3y}{2} = 2\frac{1}{2}. \end{cases}$

24. $\begin{cases} x + \dfrac{y-2}{7} + 5 = 0. \\ \dfrac{x-10}{3} + 4y = 3. \end{cases}$

28. $\begin{cases} \dfrac{5x-4y}{6} = \dfrac{2x-y}{3}. \\ 7 - \dfrac{x-y}{6} = \dfrac{x}{4} + \dfrac{y}{2}. \end{cases}$

29. $\begin{cases} \dfrac{x}{4} + 6 = \dfrac{y}{3} - 4. \\ \dfrac{x+y}{10} + \dfrac{y}{3} = \dfrac{2x-y}{4} + 25. \end{cases}$

30. $\begin{cases} \dfrac{5y}{6} - \dfrac{4y-19}{3} = \dfrac{x}{6} + \dfrac{20-2y}{3}. \\ \dfrac{x+5y}{6} + 5 = \dfrac{2y+21}{3}. \end{cases}$

39. If we have three independent equations containing three unknown quantities, we can eliminate one of the unknowns from two of the equations, thus obtaining a single equation containing two unknown quantities. In like manner, we can obtain from the third equation and either of the others a second equation containing two unknown quantities.

I. Solve $\begin{cases} 3x + 4y + 2z = 8. & (1) \\ 4x - 5y - 3z = 4. & (2) \\ 2x + 6y + 5z = 13. & (3) \end{cases}$

Multiplying (1) by 5, $15x + 20y + 10z = 40.$

Multiplying (2) by 4, $16x - 20y - 12z = 16.$

By addition, $31x - 2z = 56.$ (4)

Multiplying (1) by 3, $9x + 12y + 6z = 24.$

Multiplying (3) by 2, $4x + 12y + 10z = 26.$

By subtraction, $5x - 4z = -2.$ (5)

Multiplying (4) by 2, $62x - 4z = 112.$ (6)

Subtracting (5) from (6), $57x = 114.$

$$x = 2.$$

Substituting this value of x in (5), $10 - 4z = -2.$

$$-4z = -12.$$

$$z = 3.$$

Substituting these values of x and z in (1),

$$6 + 4y + 6 = 8.$$

$$4y = -4.$$

$$y = -1.$$

EXAMPLES.

Solve the following equations:

1. $\begin{cases} x + y + z = 18. \\ x - y + z = 8. \\ x - y - z = 2. \end{cases}$

2. $\begin{cases} x + y + z = 9. \\ x + 2y + 3z = 22. \\ 2x + 3y + 5z = 36. \end{cases}$

3. $\begin{cases} x + 4y - 3z = 19. \\ 4x - y + 5z = 25. \\ 5x + 6y - z = 53. \end{cases}$

4. $\begin{cases} x + 2y + 3z = 10. \\ 2x + 3y + 4z = 16. \\ 5x - 6y + 4z = 7. \end{cases}$

5. $\begin{cases} 6x + 2y - 3z = 15. \\ 4x + 3y - 2z = 15. \\ 5x + 4y - 7z = 15. \end{cases}$

6. $\begin{cases} 3x - 6y + 4z = 9. \\ 6x + 5y - 3z = 20. \\ 4x - y + 3z = 34. \end{cases}$

7. $\begin{cases} 5x - 2y = 41. \\ 4x + 3y + 2z = 4. \\ 3x - y - 4z = 7. \end{cases}$

8. $\begin{cases} 5x - 3y = 26. \\ 3x - 5z = 26. \\ x + y + z = 9. \end{cases}$

9. $\begin{cases} \frac{1}{4}x + \frac{1}{3}y + \frac{1}{2}z = 14. \\ \frac{2}{3}x - \frac{1}{2}y - \frac{3}{4}z = 4. \\ 2x + y - \frac{1}{3}z = 6. \end{cases}$

10. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 14. \\ \frac{1}{4}x - \frac{1}{2}y + \frac{1}{2}z = 2. \\ \frac{1}{3}x + \frac{1}{6}y - \frac{2}{3}z = 2. \end{cases}$

PROBLEMS.
LEADING TO SIMULTANEOUS EQUATIONS.

40. In many problems we have found that more than one number is required for the answer. Heretofore all such problems have been solved by the use of only one unknown quantity. Very often, however, the conditions are such that the answers can be obtained much more easily by using two or more unknown quantities. In all such cases we must be sure to obtain just as many independent equations as there are unknown quantities.

I. A merchant sold 15 yards of silk and 8 yards of cambric for \$13.20. To another customer he sold at the same rate 9 yards of silk and 5 yards of cambric for \$7.95. Find the price of each per yard.

$x =$ the number of cents for 1 yard of silk.
$y =$ the number of cents for 1 yard of cambric.

According to the conditions of the problem,

$$15x + 8y = 1320.$$
$$9x + 5y = 795.$$

Solving these equations,
$x = 80$; $y = 15$.

Ans. $\begin{cases} \text{Silk, 80 cents.} \\ \text{Cambric, 15 cents.} \end{cases}$

II. If the larger of two numbers is divided by the smaller, the quotient is 2 and the remainder is 7. The sum of the numbers is 2 more than twice the difference. Find the numbers.

$x =$ the larger number.
$y =$ the smaller number.

Since in division a remainder is indicated by writing it above the divisor,

$$\frac{x}{y} = 2 + \frac{7}{y}.$$

According to the second condition,

$$x + y = 2(x - y) + 2.$$

Solving these equations,

$x = 25$; $y = 9$. *Ans.* 9 and 25.

III. If the floor of a room were 2 feet longer and 1 foot wider, it would contain 40 square feet more; but if its length and width were each 1 foot less, it would contain 25 square feet less. Find the dimensions.

$x =$ the number of feet in length.
$y =$ the number of feet in width.
$xy =$ the number of square feet in the area.

If the length were increased by 2 feet, and the width by 1 foot, the number of square feet in the area would be $(x + 2)(y + 1)$; hence

$$(x + 2)(y + 1) = xy + 40.$$

Likewise, $$(x - 1)(y - 1) = xy - 25.$$

Solving these equations,

$x = 14$; $y = 12$. *Ans.* $\begin{cases} \text{Length, 14 feet.} \\ \text{Width, 12 feet.} \end{cases}$

IV. If the numerator of a fraction is doubled and the denominator is diminished by 1, its value becomes $\frac{3}{4}$; if 1 is added to the denominator, its value becomes $\frac{1}{3}$. Find the fraction.

$x =$ the numerator.
$y =$ the denominator.
$\frac{x}{y} =$ the fraction.

$$\frac{2x}{y - 1} = \frac{3}{4}.$$

$$\frac{x}{y + 1} = \frac{1}{3}.$$

Solving these equations,

$x = 6$; $y = 17$. *Ans.* $\frac{6}{17}$.

V. A number consisting of two digits is equal to four times the sum of the digits. If 18 is added to the number, the order of the digits is reversed. Find the number.

$$x = \text{the tens' digit.}$$
$$y = \text{the units' digit.}$$
$$10x + y = \text{the number.}$$
$$10x + y = 4(x + y).$$
$$10x + y + 18 = 10y + x.$$

Solving these equations,

$x = 2$; $y = 4$. *Ans.* 24.

NOTE. If a number consists of three digits x, y, and z, the number is denoted by $100x + 10y + z$. For example, $638 = 100 \times 6 + 10 \times 3 + 8$.

EXAMPLES.

1. If one of two numbers is multiplied by 4 and the other by 6, the sum of the products is 62; if the former is multiplied by 7 and the latter by 4, the difference of the products is 36. Find the numbers.

2. Find two numbers such that their sum divided by the larger equals $1\frac{1}{3}$, and their difference is 6 more than the smaller number.

3. The sum of the ages of a father and son is 56 years, and the father's age is 8 years more than twice the son's age. Find their ages.

4. Three times A's age is 42 years more than B's age, and one-half of B's age is 7 years less than A's age. Find their ages.

5. Three years ago a father was three times as old as his son, and eight years hence the father will be twice as old as his son. Find their ages.

6. Alfred, Henry, and James together have 95 cents. If Alfred is absent, the amount is 75 cents; if James is absent, the amount is 65 cents. How much has each?

7. If A's money were increased by $720, he would have three times as much as B; but if B's money were diminished by $100, he would have one-half as much as A. How much money has each?

8. If A gives B $300, B will then have twice as much money as A; if, however, B gives A $300, A will have five times as much as B. How much money has each?

9. A man paid a bill of $2.30 with eight pieces of silver, dimes and quarter-dollars. How many of each kind were used?

10. A man paid $136 for 20 barrels of flour, giving $8 a barrel for first quality and $5 a barrel for second quality. How many barrels were there of each kind?

11. A farmer sold 6 cows and 10 sheep for $290. To another person he sold 10 cows and 5 sheep for $390. What was the price per head of each?

12. A grocer can sell for 84 cents either 4 pounds of rice and 8 pounds of sugar, or 6 pounds of rice and 5 pounds of sugar. Find the price of each per pound.

13. The wages of 12 men and 8 boys amount to $31.20, and 5 men together receive $2.80 more than 8 boys. Find the wages of each.

14. A man and his wife working together 6 days received $16.50. At another time the man worked 8 days and the wife 4 days, and they received $18. Find the daily wages of each.

15. A grocer wishes to mix teas worth respectively 40 cents and 65 cents a pound so as to form a mixture of 100 pounds worth 50 cents a pound. How many pounds of each kind shall he take?

16. How many pounds of chicory at 6 cents a pound and coffee at 28 cents a pound must be mixed with 20 pounds of coffee worth 35 cents a pound in order to form a mixture of 75 pounds worth 24 cents a pound?

17. If the larger of two numbers is divided by the smaller, the quotient is 2 and the remainder 6. If five times the smaller is divided by the larger, the quotient is 2 and the remainder 1. Find the numbers.

18. If the sum of two numbers is divided by the smaller, the quotient is 3 and the remainder 4. Five times the smaller number is 3 more than twice the larger. Find the numbers.

19. A man has a rectangular lawn. If he adds 8 yards to the shorter side, the lawn will be a square. If he adds 8 yards to both dimensions, the area will be increased by 448 square yards. Find the dimensions of the lawn.

20. If a rectangular lot of land were 3 feet longer and 2 feet wider, it would contain 342 square feet more; if it were 2 feet longer and 3 feet wider, it would contain 360 square feet more. Find the dimensions.

21. A sum of money was divided equally among some boys. If there had been 2 more boys, each would have received 4 cents less; if there had been 2 less boys, each would have received 6 cents more. How many boys were there, and how much did each receive?

22. A fraction becomes equal to $\frac{2}{3}$ if 3 is added to its numerator, and equal to $\frac{1}{5}$ if 3 is added to its denominator. Find the fraction.

23. If 3 is added to both numerator and denominator of a fraction, its value becomes $\frac{3}{5}$; if 3 is subtracted from both numerator and denominator, its value becomes $\frac{3}{7}$. Find the fraction.

24. If the numerator of a fraction is doubled and the denominator is increased by 1, its value becomes $\frac{1}{2}$; if the denominator is doubled and the numerator is increased by 1, its value becomes $\frac{1}{6}$. Find the fraction.

25. The sum of the two digits of a number is 11. If 27 is added to the number, the order of the digits is reversed. Find the number.

26. A number consisting of two digits is equal to six times the sum of the digits. If 9 is subtracted from the number, the order of the digits is reversed. Find the number.

27. The sum of the two digits of a number is 11. If the number is divided by the sum of the digits, the quotient is 6 and the remainder 8. Find the number.

28. A number consists of three digits, the units' digit being 1. If the hundreds' and tens' digits are interchanged, the number is diminished by 360. If the hundreds' digit is divided by 3, and the tens' and units' digits are interchanged, the number is diminished by 409. Find the number.

29. The sum of the three digits of a number is 14, and the tens' digit is equal to the sum of the other two. If 99 is subtracted from the number, the order of the digits is reversed. Find the number.

30. A man has $1500 at interest. On one part he receives 3% interest, on a second part 4%, and on the third part 5%. The entire interest is $62, and the part receiving 4% interest is equal to one-half the sum of the other two. Find the three parts.

CHAPTER XII.

FACTORS.

41. An algebraic expression which will divide another without a remainder is called a **divisor** or **factor** of that expression.

The algebraic expressions which multiplied together produce a given expression are called the **factors** of that expression.

The process of resolving an algebraic expression into its factors is called **factoring.**

NOTE. It is understood that only those factors which are free from fractions and radical signs are to be considered.

The factors of a monomial can be determined by inspection. For example, the factors of $15ab^2$ are 3, 5, a, b, and b.

Many polynomials cannot be factored, but there are certain cases which always can be factored. The most important are given in this chapter.

42. CASE I. *When all the terms have a common monomial factor.*

I. Factor $15a^4 - 9a^2b - 2a^2$.

$15a^4 - 9a^2b - 2a^2 = 3a^2(5a^2 - 3b - 2)$. We see by inspection that $3a^2$ is a factor of every term. Dividing the entire expression by $3a^2$, the quotient is $5a^2 - 3b - 2$. Hence the factors are $3a^2$ and $5a^2 - 3b - 2$.

It is customary to write the factors as a product, placing each polynomial factor in parentheses. Monomial factors are grouped together, and as a whole are generally spoken of as *the monomial factor.*

EXAMPLES.

Factor the following:

1. $a^3 - 7a$.
2. $3a + 3ab$.
3. $b^4 - b^3$.
4. $3x^2 - 6x$.
5. $12 - 6c^2$.
6. $8c^2d^2 - 24cd^3$.
7. $28m^3n^2 - 8mn^6$.
8. $72y^3z^2 - 32y^4z^3$.
9. $ac^2 - cx^2 + cy$.
10. $3x^3 - 2x^2 + x$.
11. $4m^3 - 2m^2n + mn^2$.
12. $5y - 10y^2 + 15y^4$.
13. $3a^2b - 6ab^2 + 3ab$.
14. $a^4b^3c^2 + a^3b^2c^3 + a^2b^3c^4$.
15. $24x^4y - 36x^3y^2 + 12x^2y^3$.
16. $18c^6d^4 + 42c^4d^3 - 24c^2d^2$.

43. CASE II. *When the terms can be arranged in groups, each of which can be factored as in Case* I.

I. Factor $a^2 - ab - ac + bc$.

$$\begin{aligned} a^2 - ab - ac + bc &= (a^2 - ab) - (ac - bc) \\ &= a(a - b) - c(a - b) \\ &= (a - b)(a - c). \end{aligned}$$

We see by inspection that a is a factor of the first two terms, and c of the last two terms. Hence we divide the polynomial into two groups and factor each group as in Case I. We then see that $a - b$ is a factor of each group. Dividing the entire expression by $a - b$, we obtain $a - c$ as the other factor.

EXAMPLES.

Factor the following:

1. $ax + bx + ay + by$.
2. $px - py + qx - qy$.
3. $m^2 - mx - my + xy$.
4. $x^3 + x^2 - x - 1$.
5. $x^3 + 2x^2 + 4x + 8$.
6. $a^2 - ab - 4a + 4b$.
7. $16 - 4a + 4b - ab$.
8. $7c + cd - 7d - d^2$.

9. $3 + 4m - 6m^2 - 8m^3$. **11.** $h^4 - 2h^3 + 3h - 6$.

10. $4a^3 - 6a^2 - 6a + 9$. **12.** $c^3 + 6c^2d + cd^2 + 6d^3$.

13. $m^2n^2 - mnp - mnq + pq$.

14. $2am + 2bn + a^2m + abn$.

44. CASE III. *When a trinomial is a perfect square.*

I. Factor $a^2 + 2ab + b^2$.

$a^2 + 2ab + b^2 = (a + b)^2$. In § 30 we learned that $(a + b)^2 = a^2 + 2ab + b^2$. Writing this identity in reverse order, the factors of $a^2 + 2ab + b^2$ are $a + b$ and $a + b$.

A trinomial which is a perfect square must satisfy the following test: two terms are squares and positive, and the third term is twice the product of the square roots of the other two.

NOTE. To find the square root of a monomial, *find the square root of the coefficient, and divide the exponent of each letter by* 2.

II. Factor $16x^2 - 40xy^2 + 25y^4$.

$16x^2 - 40xy^2 + 25y^4 = (4x - 5y^2)^2$. This trinomial satisfies the test of a perfect square. The square root of $16x^2$ is $4x$, and the square root of $25y^4$ is $5y^2$; connect these terms by the sign —, the sign of the middle term. Then $16x^2 - 40xy^2 + 25y^4$ is the square of $4x - 5y^2$.

EXAMPLES.

Factor the following:

1. $x^2 - 2xy + y^2$.

2. $a^2 + 2a + 1$.

3. $x^2 - 2mnx + m^2n^2$.

4. $a^2b^2 + 2abcd + c^2d^2$.

5. $a^2 + 6a + 9$.

6. $c^2d^2 - 14cd + 49$.

7. $a^2c^2 - 10acd^2 + 25d^4$.

8. $x^6 + 12x^3y^3 + 36y^6$.

9. $4a^2 - 4a + 1$.

10. $9m^4 + 42m^2n + 49n^2$.

11. $16p^2 + 8pq + q^2$.

12. $16a^4 - 56a^2b^2 + 49b^4$.

13. $100a^2 - 20a + 1$.

14. $64h^2 + 48hk + 9k^2$.

15. $81p^2 + 36pq + 4q^2$.

16. $36 - 132c^4 + 121c^8$.

45. CASE IV. *When a binomial is the difference of two squares.*

I. Factor $a^2 - b^2$.

$a^2 - b^2 = (a + b)(a - b)$. In § 30 we learned that $(a+b)(a-b) = a^2 - b^2$. Writing this identity in reverse order, the factors of $a^2 - b^2$ are $a + b$ and $a - b$.

The difference of two squares can be resolved into two binomial factors, the sum of the square roots of the terms and the difference of the same square roots.

II. Factor $16x^4 - 25y^2$.

$16x^4 - 25y^2 = (4x^2 + 5y)(4x^2 - 5y)$. The square root of $16x^4$ is $4x^2$, and the square root of $25y^2$ is $5y$. Hence the factors of $16x^4 - 25y^2$ are $4x^2 + 5y$ and $4x^2 - 5y$.

EXAMPLES.

Factor the following:

1. $c^2 - d^2$.

2. $x^2y^2 - z^2$.

3. $a^2 - 1$.

4. $y^2 - 25$.

5. $m^2 - 4n^2$.

6. $16 - a^2$.

7. $4p^2 - 9q^2$.

8. $25a^2 - 49b^2$.

9. $121 - 4b^2$.

10. $a^2b^2c^2 - 9$.

11. $81a^2b^2 - 1$.

12. $d^4 - 144$.

13. $1 - 36z^4$.

14. $x^6 - y^4$.

15. $49x^4 - 16z^4$.

46. Some polynomials include three terms which form a perfect square. Many of them can be expressed as the difference of two squares, and then factored by the principle of Case IV.

I. Factor $a^2 - 2ab + b^2 - c^2$.

$$\begin{aligned} a^2 - 2ab + b^2 - c^2 &= (a^2 - 2ab + b^2) - c^2 \\ &= (a-b)^2 - c^2 \\ &= (a-b+c)(a-b-c). \end{aligned}$$

II. Factor $x^2 - y^2 + 2yz - z^2$.

$$\begin{aligned} x^2 - y^2 + 2yz - z^2 &= x^2 - (y^2 - 2yz + z^2) \\ &= x^2 - (y-z)^2 \\ &= (x+y-z)(x-y+z). \end{aligned}$$

EXAMPLES.

Factor the following:

1. $(a+b)^2 - c^2$.
2. $(x-y)^2 - 4$.
3. $a^2 - (m+n)^2$.
4. $x^2 - (y-z)^2$.
5. $b^2 + 2bc + c^2 - d^2$.
6. $m^2 - n^2 - 2np - p^2$.
7. $h^2 - 2hk + k^2 - m^2$.
8. $p^2 - x^2 + 2xy - y^2$.
9. $a^2 - 4a + 4 - 25b^2$.
10. $a^2 + 6ac + 9c^2 - 49d^2$.
11. $4a^2 - b^2 + 6bc - 9c^2$.
12. $49 - x^2 - 10xy - 25y^2$.
13. $4x^2 - 9y^2 - 24yz - 16z^2$.
14. $36x^2 - 36xy + 9y^2 - 4z^2$.

47. CASE V. *When a binomial is the sum or difference of two cubes.*

In § 37 we learned that the sum of two cubes can be divided by the sum of their cube roots, and the difference of two cubes can be divided by the difference of their cube roots. In either case the second factor can be determined by the principles given in § 37.

I. Factor $a^3 + b^3$.

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$.

Since $(a^3 + b^3) \div (a+b) = a^2 - ab + b^2$, the factors of $a^3 + b^3$ are $a+b$ and $a^2 - ab + b^2$.

II. Factor $8x^3 - 27y^3$.

$8x^3 - 27y^3 = (2x - 3y)(4x^2 + 6xy + 9y^2)$. The cube root of $8x^3$ is $2x$, and the cube root of $27y^3$ is $3y$. Hence the factors of $8x^3 - 27y^3$ are $2x - 3y$ and $4x^2 + 6xy + 9y^2$.

NOTE. To find the cube root of a monomial, *find the cube root of the coefficient, and divide the exponent of each letter by* 3.

EXAMPLES.

Factor the following:

1. $c^3 + d^3$.	6. $1 + 8a^3c^3$.	11. $x^6 + y^6$.
2. $m^3 - n^3$.	7. $216 - z^3$.	12. $x^6 - y^6$.
3. $x^3 + 1$.	8. $x^3y^3 + 343$.	13. $a^{12} + b^{12}$.
4. $y^3 - 1$.	9. $27a^3 + 125x^3$.	14. $m^6 + 512$.
5. $27m^3 + n^3$.	10. $64c^3 - 343d^3$.	15. $125y^9 + 1$.

48. CASE VI. *When a trinomial has the form* $x^2 + ax + b$.

In § 31 we have four cases of multiplication of binomials where the products are of the form under consideration. Writing these in reverse order, we have the following:

$$x^2 + 10x + 21 = (x + 7)(x + 3).$$
$$x^2 - 10x + 21 = (x - 7)(x - 3).$$
$$x^2 + 4x - 21 = (x + 7)(x - 3).$$
$$x^2 - 4x - 21 = (x - 7)(x + 3).$$

We see that each of these trinomials can be resolved into two binomial factors. The first term of both factors is the square root of the first term of the trinomial, and the second terms of the factors are two numbers whose *product* is the last term of the trinomial and whose *algebraic sum* is the coefficient of the second term of the trinomial.

I. Factor $x^2 + 11x + 24$.

$x^2 + 11x + 24 = (x + 8)(x + 3)$. We wish to find two numbers whose product is 24 and whose sum is 11. To obtain the product 24, we may take 24 and 1, 12 and 2, 8 and 3, or 6 and 4; from these we select 8 and 3 as the pair whose sum is 11. Hence the factors are $x + 8$ and $x + 3$.

II. Factor $x^2 - 17ax + 16a^2$.

$x^2 - 17ax + 16a^2 = (x - 16a)(x - a)$. We wish to find two quantities whose product is $16a^2$ and whose sum is $-17a$. Since the sum is negative and the product positive, the quantities must both be negative. By inspection we find that the quantities are $-16a$ and $-a$. Hence the factors are $x - 16a$ and $x - a$.

III. Factor $a^2 + 7a - 18$.

$a^2 + 7a - 18 = (a + 9)(a - 2)$. We wish to find two numbers whose product is -18 and whose sum is 7. Since the product is negative, the numbers have unlike signs; since the sum is positive, the larger number is positive. By inspection we find that the two numbers are 9 and -2. Hence the factors are $a + 9$ and $a - 2$.

IV. Factor $m^2 - mx - 20x^2$.

$m^2 - mx - 20x^2 = (m - 5x)(m + 4x)$. We wish to find two quantities whose product is $-20x^2$ and whose sum is $-x$. Since the product is negative, the quantities have unlike signs; since the sum is negative, the larger quantity is negative. By inspection we find that the quantities are $-5x$ and $4x$. Hence the factors are $m - 5x$ and $m + 4x$.

EXAMPLES.

Factor the following:

1. $x^2 + 5x + 6$.
2. $x^2 - 5x + 6$.
3. $y^2 - 6y + 8$.
4. $y^2 + 7y - 8$.
5. $a^2 + 7a + 10$.
6. $b^2 - 12b + 11$.

7. $z^2 + 7z + 12$.
8. $z^2 + 4z - 12$.
9. $m^2 + m - 20$.
10. $m^2 - 8m - 20$.
11. $c^2 - 17c + 30$.
12. $d^2 + 15d + 36$.
13. $h^2 + 11h - 42$.
14. $p^2 - 18p + 56$.
15. $q^2 - q - 72$.
16. $x^2 - ax - 2a^2$.
17. $x^2 - 5xy + 4y^2$.
18. $a^2 + 7ab + 6b^2$.
19. $a^2 - 8ac + 12c^2$.
20. $c^2 + 2cd - 15d^2$.
21. $m^2 - 3mx - 18x^2$.
22. $m^2 + 11mn + 18n^2$.
23. $1 - 10n + 21n^2$.
24. $1 + 9p - 22p^2$.
25. $x^2 - 18bx + 32b^2$.
26. $y^2 + 6cy - 40c^2$.
27. $c^2 - 13cy - 48y^2$.
28. $a^2 + 11az - 60z^2$.
29. $z^2 - 12az - 64z^2$.
30. $p^2 + 19pq + 84q^2$.

49. Frequently more than one principle of factoring can be applied to a given expression. If all the terms of an expression have a common monomial factor, this factor should first be placed before parentheses as shown in Case I. If an expression can be expressed as the difference of two squares, it should be resolved as in Case IV. before applying any other principle.

In addition to the six cases that have been given, trinomials of the form $a^4 + a^2b^2 + b^4$ should be mentioned. By multiplying $a^2 + ab + b^2$ by $a^2 - ab + b^2$, we find that $a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2)$.

I. Factor $3x^4y - 6x^3y - 72x^2y$.

$$\begin{aligned} 3x^4y - 6x^3y - 72x^2y &= 3x^2y(x^2 - 2x - 24) \quad \ldots\ldots \text{ Case I.} \\ &= 3x^2y(x-6)(x+4) \quad \ldots\ldots \text{ Case VI.} \end{aligned}$$

II. Factor $a^6 - b^6$.

$$a^6 - b^6 = (a^3 + b^3)(a^3 - b^3) \quad \ldots\ldots \text{ Case IV.}$$
$$= (a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2) \quad \ldots \text{ Case V.}$$

III. Factor $a^2x - a^2 - x + 1$.

$$a^2x - a^2 - x + 1 = a^2(x - 1) - (x - 1)$$
$$= (x - 1)(a^2 - 1) \quad \ldots\ldots \text{ Case II.}$$
$$= (x - 1)(a + 1)(a - 1) \quad \ldots\ldots \text{ Case IV.}$$

MISCELLANEOUS EXAMPLES.

1. $a^2 - 26\,ab + 88\,b^2$.
2. $x^2 - 10\,ax - 96\,a^2$.
3. $x^4 - x^2 - 9$.
4. $a^2 + ax - ay - xy$.
5. $a^2b - ab + a^2 - a$.
6. $a^2 - 16\,a + 64$.
7. $a^4 - b^4$.
8. $a^6 - 1$.
9. $a^6 + b^6$.
10. $x^2 - x^8$.
11. $b^8 - c^8$.
12. $a^2 - b^2 - c^2 - 2\,bc$.
13. $x^2 + y^2 - z^2 - 2\,xy$.
14. $x^4 + x^2y^2 + y^4$.
15. $a^3 - 9\,ab^2$.
16. $16\,x^4 - 9\,x^2z^2$.
17. $c^5 + 8\,c^2d^3$.
18. $5\,x^4y^2 - 3\,x^3y + 5\,x^2$.
19. $a^4 + 6\,a^2b^2c^2 + 9\,b^4c^4$.
20. $a^4 - a^3 + 8\,a - 8$.
21. $6\,m^4 - 18\,m^3 - 60\,m^2$.
22. $b^3 - 2\,b^2 - 4\,b + 8$.
23. $a^4 - 8\,a^3 + 16\,a^2$.
24. $4\,y^6 + 4\,y^3$.
25. $9\,x^4 - 16\,x^2y^2$.
26. $64 - y^6$.
27. $m^4 + 2\,m^2 + 1$.
28. $x^6 + 7\,x^3 - 8$.
29. $x^2 - y^2 - 4\,yz - 4\,z^2$.
30. $a^3x + a^3y - b^3x - b^3y$.
31. $16\,x^4 - 1$.
32. $3\,x^7 - 3\,xy^6$.
33. $x^5 + x^3 + x$.
34. $x^4 + x^3y + xy^3 + y^4$.
35. $36\,a^2 + 84\,a + 49$.
36. $ac^2 - bc^2 + 3\,a - 3\,b$.
37. $8\,x^2 + 24\,xy + 18\,y^2$.
38. $x^4 + 8\,x^3 - 48\,x^4$.
39. $9\,a^2 - 6\,ac + c^2 - 16\,d^2$.
40. $4\,a^3 - 3\,a^2 - 16\,a + 12$.

CHAPTER XIII.

COMMON FACTORS AND MULTIPLES.

50. A **common factor** of two or more algebraic expressions is an expression which will divide each of them without a remainder.

The **highest common factor** of two or more algebraic expressions is the expression of highest degree which will divide each of them without a remainder. For example, $2a$ and $3b$ are common factors of $12a^2b^3$ and $18a^3b$, but $6a^2b$ is the highest common factor.

From this definition it follows that the highest common factor of two or more algebraic expressions must consist of all the factors common to the expressions.

For convenience H.C.F. is used to denote the highest common factor.

NOTE. *The names greatest common divisor* (G.C.D.), *highest common divisor* (H.C.D.), and *greatest common measure* (G.C.M.) have the same meaning as *highest common factor.*

I. Find the highest common factor of $24a^5b^3c$, $32a^3b^2c^2$, and $60a^4b$.

$$24a^5b^3c = 2^3 \times 3a^5b^3c.$$
$$32a^3b^2c^2 = 2^5a^3b^2c^2.$$
$$60a^4b = 2^2 \times 3 \times 5a^4b.$$
$$\text{H. C. F.} = 2^2a^3b = 4a^3b.$$

2, a, and b are the common factors. 2 occurs at least twice in each expression, and hence must occur twice in the H.C.F. Likewise, a must occur three times in the H.C.F. Then the product of all the common factors is $4a^3b$.

NOTE. The numerical part of the highest common factor can generally be determined by inspection; in all such cases it is not necessary to write down the factors of the numerical coefficients. In this particular example we can see by inspection that 4 is the largest number that can be divided into 24, 32, and 60 without a remainder.

II. Find the highest common factor of $x^2 - 4y^2$, $x^2 - 4xy + 4y^2$, and $x^2 + xy - 6y^2$.

$$x^2 - 4y^2 = (x + 2y)(x - 2y).$$
$$x^2 - 4xy + 4y^2 = (x - 2y)^2.$$
$$x^2 + xy - 6y^2 = (x + 3y)(x - 2y).$$
$$\text{H.C.F.} = x - 2y.$$

III. Find the highest common factor of $2a^3 + 2a^2$, $4a^5 - 4a^3$, and $6a^6 + 6a^3$.

$$2a^3 + 2a^2 = 2a^2(a + 1).$$
$$4a^5 - 4a^3 = 4a^3(a^2 - 1) = 4a^3(a + 1)(a - 1).$$
$$6a^6 + 6a^3 = 6a^3(a^3 + 1) = 6a^3(a + 1)(a^2 - a + 1).$$
$$\text{H.C.F.} = 2a^2(a + 1).$$

To find the highest common factor of two or more algebraic expressions, *factor the expressions, and find the product of all the common factors, taking the lowest power of each factor.*

EXAMPLES.

Find the highest common factor of

1. $2a^4b^2$ and $3a^3b^4$.
2. $6x^2y$ and $15xy^2$.
3. $12a^3b^2c$ and $28ac^2d^3$.
4. $45x^5y^3z$ and $75x^3y^3$.
5. $16c^2d^4$, $40c^3d^6$, and $56c^4d^4$.
6. $24m^3n^6$, $48m^2n^4$, and $84m^5n$.
7. $27m^2np^3q^3$, $36n^4p^3q$, and $54np^2q^2$.
8. $28a^5m^4x$, $42a^4n^4y$, and $63a^3p^3z$.
9. $35a^5b^5c^2$, $48a^3b^6c^4$, and $60a^6b^4c^3$.
10. $39a^3x^2y$, $52b^3xy^2$, and $91c^3x^3y^3$.
11. $x^2 - xy$ and $x^2 - y^2$.
12. $c^2 - 2cd$ and $cd - 2d^2$.
13. $a^3 + a^2b$ and $a^3 + b^3$.

14. $15x - 5$ and $27x^2 - 3$.

15. $10(a + b)^3$ and $6(a + b)^2$.

16. $8(a - b)^2$ and $12(a^2 - b^2)$.

17. $x^2 - 4$ and $x^2 + 5x + 4$.

18. $m^3 + 4m^2n$ and $m^3 + 64n^3$.

19. $x^2 - 5x + 6$ and $x^2 + x - 12$.

20. $x^2 - 6x + 9$ and $x^2 + 3x - 18$.

21. $4y^3 - 4$ and $6y^2 + 12y - 18$.

22. $4a^4b - 36a^2b$ and $6a^3b^2 - 18a^2b^2 - 108ab^2$.

23. $a^2 - ab - ac + bc$ and $a^3 - c^3$.

24. $x^2 + y^2 - z^2 + 2xy$ and $x^2 - y^2 - z^2 - 2yz$.

25. $a^2 - 6a$, $a^2 - 12a + 36$, and $a^2 - 36$.

26. $2a^2b(a + b)$, $3a^3c(a + b)^2$, and $6a^4d(a + b)^3$.

27. $x^3 - y^3$, $x^4 + x^2y^2 + y^4$, and $x^6 - y^6$.

28. $a^3 - 3a^2 - 10a$, $a^4 - 9a^3 + 20a^2$, and $a^5 + a^4 - 30a^3$.

29. $a^3 - 27b^3$, $a^3 - 6a^2b + 9ab^2$, and $a^3 + a^2b - 12ab^2$.

30. $3c^4 - 6c^3 - 4c^2 + 8c$, $6c^4 - 27c^2$, and $c^3 + 6c^2 - 16c$.

51. A **multiple** of an algebraic expression is an expression which can be divided by it without a remainder.

A **common multiple** of two or more algebraic expressions is an expression which can be divided by each of them without a remainder.

The **lowest common multiple** of two or more algebraic expressions is the expression of lowest degree which can be divided by each of them without a remainder. For example, $24a^5b^6$ is a common multiple of $4a^3b^2$ and $6a^2b^4$, but $12a^3b^4$ is the lowest common multiple.

From the definition it follows that the lowest common multiple of two or more algebraic expressions must consist of all the different factors of the expressions.

For convenience L.C.M. is used to denote the lowest common multiple.

I. Find the lowest common multiple of $14a^2b^5c$, $24a^4bd^3$, and $36a^6b^2$.

$$\begin{aligned} 14a^2b^5c &= 2 \times 7a^2b^5c. \\ 24a^4bd^3 &= 2^3 \times 3a^4bd^3. \\ 36a^6b^2 &= 2^2 \times 3^2a^6b^2. \\ \text{L.C.M.} &= 2^3 \times 3^2 \times 7a^6b^5cd^3 \\ &= 504a^6b^5cd^3. \end{aligned}$$

The lowest common multiple not only must contain all the different factors, but also each factor must be present as many times as it is found in any one expression. Then the product of all the different factors is $504a^6b^5cd^3$.

NOTE. The numerical part of the lowest common multiple can frequently be determined by inspection ; in such cases it is not necessary to write down the factors of the numerical coefficients.

II. Find the lowest common multiple of $6x^3 - 12x^2y$, $3x^3 - 3x^2y - 6xy^2$, and $2x^3 + 4x^2y + 2xy^2$.

$$6x^3 - 12x^2y = 6x^2(x - 2y).$$
$$3x^3 - 3x^2y - 6xy^2 = 3x(x^2 - xy - 2y^2) = 3x(x - 2y)(x + y).$$
$$2x^3 + 4x^2y + 2xy^2 = 2x(x^2 + 2xy + y^2) = 2x(x + y)^2.$$
$$\text{L.C.M.} = 6x^2(x - 2y)(x + y)^2.$$

To find the lowest common multiple of two or more algebraic expressions, *factor the expressions, and find the product of all the different factors, taking the highest power of each factor.*

EXAMPLES.

Find the lowest common multiple of

1. $a^5b^6c^3$ and $3a^2b^2c^2$.
2. $4x^3y^2$ and $6x^2y^3$.
3. $8a^2b^4c^6$ and $20a^4b$.
4. $15x^4y^2z$ and $40x^2z^2$.
5. a^2bc, ab^2c, and abc^2.
6. $2x^3y$, $3x^2z$, and $4y^2z^2$.

7. $12\,m^4n^6$, $24\,m^2n^4$, and $30\,m^3n^3$.

8. $18\,m^2np^3$, $27\,n^3n^3q^5$, and $81\,m^7p^2q^3$.

9. $22\,a^4x^2y$, $25\,b^4x^3y$, and $33\,c^4x^4z$.

10. $30\,x^2y^3z$, $35\,x^4y^2z$, and $42\,xy^2z^3$.

11. $x^2 - y^2$ and $x^2 - 2\,xy + y^2$.

12. $a^2 - b^2$ and $a^3 - b^3$.

13. $m^3 + 3\,m^2$ and $(m + 3)^3$.

14. $4\,a^2 - 4$ and $2\,a^2 + 2\,a$.

15. $(x - y)^3$ and $x^3 - y^3$.

16. $x^3y + xy^3$ and $xz + yz$.

17. $k^2 + k - 12$ and $k^2 - k - 20$.

18. $y^2 + 9\,yz + 8\,z^2$ and $y^2 - 5\,yz - 6\,z^2$.

19. $ax - bx - ay + by$ and $a^3x - a^3y$.

20. $x^4 + x^2y^2 + y^4$ and $x^4 + xy^3$.

21. xy, $x\,(x - y)$, and $y\,(x^2 - y^2)$.

22. $a^4\,(x + y)$, $a^3\,(x + y)^2$, and $a^2\,(x + y)^3$.

23. $15\,(a - b)^2$, $25\,(a - b)^3$, and $35\,(a - b)^4$.

24. $a\,(m + n)$, $a^2\,(m - n)$, and $a^2b^2\,(m^2 - n^2)$.

25. $(c + d)^2$, $(c - d)^2$, and $c^2 - d^2$.

26. $am^2 - an^2$, $m^4 - mn^3$, and $m^3n + n^4$.

27. $p^2 - q^2$, $p^2 + q^2$, and $p^4 - q^4$.

28. $c^3 - c$, $c^3 + c^2 + c + 1$, and $c^3 - c^2 + c - 1$.

29. $2\,x^2 - 2\,x - 24$, $3\,x^2 - 18\,x + 24$, and $4\,x^2 + 4\,x - 24$.

30. $x^2 - (y + z)^2$, $y^2 - (x + z)^2$, and $z^2 - (x + y)^2$.

CHAPTER XIV.

FRACTIONS.

52. We have learned that the operation of division can be indicated in the fractional form. For example, the quotient obtained by dividing a by b is written $\frac{a}{b}$. When we consider this expression as a whole, we call it a **fraction.** The dividend a is called the **numerator,** and the divisor b the **denominator.** The two together are called the **terms** of the fraction.

If the same factor is inserted in both numerator and denominator of a fraction, the effect is the same as multiplying the fraction by 1, which does not affect the value of the fraction. Likewise, removing the same factor from both numerator and denominator is the same as dividing the fraction by 1, and the value of the fraction remains the same. This important principle may be stated as follows:

Multiplying or dividing both numerator and denominator of a fraction by the same factor does not change the value of the fraction.

REDUCTION OF FRACTIONS TO LOWEST TERMS.

53. A fraction is in its **lowest terms** when the numerator and denominator have no common factor.

The factors common to numerator and denominator can be removed by dividing both terms by their highest common factor.

I. Reduce $\frac{30\,a^3b^4c^2}{48\,a^2b^2c^5}$ to its lowest terms.

$\frac{30\,a^3b^4c^2}{48\,a^2b^2c^5}=\frac{5\,ab^2}{8\,c^3}$ By inspection we see that $6\,a^2b^2c^2$ is the highest common factor of numerator and denominator. By dividing both terms by $6\,a^2b^2c^2$ the fraction is reduced to its lowest terms.

II. Reduce $\frac{x^2-3\,x-10}{x^3+8}$ to its lowest terms.

$\frac{x^2-3\,x-10}{x^3+8}=\frac{(x-5)\,(x+2)}{(x+2)(x^2-2\,x+4)}=\frac{x-5}{x^2-2\,x+4}$ Factoring both numerator and denominator, we see that $x+2$ is common to both. By removing this factor the fraction is reduced to its lowest terms.

Note. If all the factors of the numerator are removed, the resulting fraction has 1 for its numerator. If all the factors of the denominator are removed, it becomes a case of exact division and the denominator vanishes.

EXAMPLES.

Reduce the following fractions to their lowest terms:

1. $\frac{12\,x^3}{28\,x^2}$.
2. $\frac{8\,ax}{4\,a^2x}$.
3. $\frac{7\,mxy}{9\,abm}$.
4. $\frac{28\,a^4b^2}{36\,a^4c^2}$.
5. $\frac{12\,c^3d}{52\,c^4d^4}$.
6. $\frac{9\,a^2mn^2}{15\,am^2n}$.
7. $\frac{12\,mnp^2}{27\,mnp^2}$.
8. $\frac{15\,a^3x^4y^2}{35\,a^3x^4y^2}$.
9. $\frac{140\,x^3y^2z}{35\,x^2y^2z^2}$.
10. $\frac{30\,ab^2cd^3}{36\,a^2b^2cd^2}$.
11. $\frac{b^2}{ab+b^2}$.
12. $\frac{mx-my}{mz}$.
13. $\frac{a+b}{a^2-b^2}$.
14. $\frac{a-b}{a^3-b^3}$.
15. $\frac{bc-b}{bc^2-b}$.
16. $\frac{m^2-n^2}{m^3-n^3}$.
17. $\frac{3\,m^2-3}{m^2+2\,m+1}$.
18. $\frac{3\,a^2-4\,ac}{9\,a^2-16\,c^2}$.

19. $\dfrac{p^3 - pq^2}{p^2 - 2pq + q^2}$.

20. $\dfrac{a^2 + 2ab + b^2}{a^2 - 3ab - 4b^2}$.

21. $\dfrac{16x^2 - 9y^2}{16x^2 + 24xy + 9y^2}$.

22. $\dfrac{c^4 + c^3 - 20c^2}{c^3 - c^2 + 30c}$.

23. $\dfrac{6a^4x - 6x^5}{9a^2y + 9x^2y}$.

24. $\dfrac{a^2 + ab + a + b}{a^2 - b^2}$.

25. $\dfrac{a^4 - b^4}{a^6 - b^6}$.

26. $\dfrac{(x + y)^2 - z^2}{xz + yz + z^2}$.

REDUCTION OF FRACTIONS TO INTEGRAL OR MIXED EXPRESSIONS.

54. An **integral expression** is an expression which has no fractional part; as $3ac$ and $2a - b$.

NOTE. An integral expression may be treated as a fraction whose denominator is 1; thus, $a - b$ is the same as $\dfrac{a-b}{1}$.

A **mixed expression** is an expression consisting of an integral expression and a fraction; as $2a + \dfrac{b}{c}$.

I. Reduce $\dfrac{6a^2 - 9a + 4}{3a}$ to a mixed expression.

$$\frac{6a^2 - 9a + 4}{3a} = 2a - 3 + \frac{4}{3a}$$

This is merely an example in division where there is a remainder. The quotient is $2a - 3$, and the remainder is 4, which is written above the divisor in fractional form.

II. Reduce $\dfrac{3x^2 - 14x + 5}{x - 4}$ to a mixed expression.

$$\begin{array}{r|l}
x-4)\,3x^2 - 14x + 5 & (3x - 2 \\
3x^2 - 12x & \\
\hline
-\ 2x + 5 & \\
-\ 2x + 8 & \\
\hline
-3 &
\end{array}$$

The quotient is $3x - 2$ and the remainder is -3. The fraction must have the same sign as the remainder.

Ans. $3x - 2 - \dfrac{3}{x-4}$.

NOTE. When there is no remainder, the answer is an integral expression.

EXAMPLES.

Reduce to integral or mixed expressions

1. $\dfrac{28xy+6z}{7}$.
2. $\dfrac{20a^2-5b}{4a}$.
3. $\dfrac{3a^2x-ax^3}{ax}$.
4. $\dfrac{3ab+4ac+5d}{a}$.
5. $\dfrac{3ab+2b^2}{a+b}$.
6. $\dfrac{a^2+b^2}{a+b}$.
7. $\dfrac{a^2-2ab+2b^2}{a-b}$.
8. $\dfrac{x^2-3x+2}{x-2}$.
9. $\dfrac{x^3-1}{x+1}$.
10. $\dfrac{2a^2-2b^2-4}{a-b}$.
11. $\dfrac{x^2+xy+y^2}{x^2-y^2}$.
12. $\dfrac{x^3-x^2-x+1}{x^2+x+1}$.

REDUCTION OF INTEGRAL AND MIXED EXPRESSIONS TO FRACTIONS.

55. I. Reduce $a+b$ to a fraction whose denominator is c.

$a+b=\dfrac{ac+bc}{c}$ Considering $a+b$ as the fraction $\dfrac{a+b}{1}$, we can multiply both numerator and denominator by c and thus obtain the desired result.

In any case an integral expression is equal to a fraction whose numerator is the product of the integral expression and the denominator.

II. Reduce $x-y+\dfrac{x^2-2y^2}{x+y}$ to a fractional form.

$$x-y+\frac{x^2-2y^2}{x+y}=\frac{x^2-y^2+x^2-2y^2}{x+y}=\frac{2x^2-3y^2}{x+y}$$

$x-y=\dfrac{x^2-y^2}{x+y}$; adding $\dfrac{x^2-2y^2}{x+y}$ to this, the numerators can be written over the common denominator. Then by combining, we obtain the desired result.

III. Reduce $x - y - \dfrac{x^2 - 2y^2}{x + y}$ to a fractional form.

$$x - y - \frac{x^2 - 2y^2}{x + y} = \frac{x^2 - y^2 - x^2 + 2y^2}{x + y} = \frac{y^2}{x + y}$$

When a fraction is preceded by the sign —, the sign of every term of the numerator is changed just as in removing parentheses.

EXAMPLES.

Reduce to a fractional form

1. $a + \dfrac{1}{b}$.
2. $2a + \dfrac{3b}{c}$.
3. $1 - \dfrac{2x}{7}$.
4. $\dfrac{1}{a} + b$.
5. $\dfrac{a}{b} - c$.
6. $b + \dfrac{b + c}{c}$.
7. $x + \dfrac{c - d}{m}$.
8. $3y - \dfrac{2y + 7z}{5}$.
9. $5a - \dfrac{2b - 3a}{4}$.
10. $1 + \dfrac{1}{x + 1}$.
11. $1 - \dfrac{p - 3q}{p + 3q}$.
12. $\dfrac{2a + b}{2a - b} + 1$.
13. $m - n + \dfrac{n^2}{m + n}$.
14. $p - q + \dfrac{p^2 - q^2}{p + q}$.
15. $a + b + \dfrac{a^2 + b^2}{a + b}$.
16. $\dfrac{a^2 - b^2}{b} + a - b$.
17. $\dfrac{x + 2}{x + 3} + x - 1$.

$\dfrac{x - 2}{x - 3} - x + 1$.

19. $x - 5 - \dfrac{x^2 - 12}{x + 2}$.
20. $a + b - \dfrac{a^3 - b^3}{a^2 - ab + b^2}$.

LOWEST COMMON DENOMINATOR.

56. When several fractions have the same denominator, they are said to have a **common denominator.** If the denominators of two or more fractions are not the same, it is always possible to reduce the fractions to equivalent fractions having for their common denominator a common multiple of the denominators. The lowest common multiple of the denominators is called the **lowest common denominator.**

For convenience L.C.D. is used to denote the lowest common denominator.

I. Reduce $\frac{bc}{2a^2x}$, $\frac{2mn}{3a^2x^3}$, and $\frac{3yz}{4a^4x^2}$ to equivalent fractions having the lowest common denominator.

$$\frac{bc}{2a^2x}=\frac{6a^2bcx^2}{12a^4x^3}$$

$$\frac{2mn}{3a^2x^3}=\frac{8a^2mn}{12a^4x^3}$$

$$\frac{3yz^2}{4a^4x^2}=\frac{9xyz^2}{12a^4x^3}$$

We find the L.C.M. of $2a^2x$, $3a^2x^3$, and $4a^4x^2$ to be $12a^4x^3$. $2a^2x$ must be multiplied by $6a^2x^2$ to obtain $12a^4x^3$; hence the numerator must be multiplied by $6a^2x^2$ in order to keep the fraction of the same value. The other two fractions are treated in the same manner.

II. Reduce $\frac{2x}{x^2-6x+9}$ and $\frac{3x}{x^2-2x-3}$ to equivalent fractions having the lowest common denominator.

$$\frac{2x}{x^2-6x+9}=\frac{2x}{(x-3)^2}=\frac{2x(x+1)}{(x-3)^2(x+1)}.$$

$$\frac{3x}{x^2-2x-3}=\frac{3x}{(x-3)(x+1)}=\frac{3x(x-3)}{(x-3)^2(x+1)}.$$

We find the L. C. D. to be $(x-3)^2(x+1)$. The first denominator must be multiplied by $x+1$ and the second by $x-3$ to obtain the L. C. D. Hence the first numerator must be multiplied by $x+1$ and the second by $x-3$ in order to keep the fractions of the same value.

To reduce fractions to equivalent fractions having the lowest common denominator, ***in each fraction multiply both numerator and denominator by the quotient obtained by dividing the lowest common denominator by the denominator of that fraction.***

NOTE. Every fraction should be in its lowest terms before finding the lowest common denominator.

EXAMPLES.

Reduce to equivalent fractions having the lowest common denominator

1. $\frac{3ab}{4}, \frac{5ac}{6}$, and $\frac{8bc}{9}$.
2. $\frac{m}{x^2}, \frac{n}{y^2}$, and $\frac{p}{xy}$.
3. $\frac{6x}{ab}, \frac{5y}{a^2}$, and $\frac{4}{ab^2c}$.
4. $\frac{2a}{3x^2}, \frac{2b}{5y^2}$, and $\frac{c}{2z}$.
5. $\frac{a-3b}{3a^2c}$ and $\frac{2a+3c}{4ab^2}$.
6. $\frac{4}{x+3}, \frac{5}{x-3}$, and $\frac{8}{x}$.
7. $\frac{1}{x}, \frac{x}{x+y}$, and $\frac{y}{x-y}$.
8. $\frac{a}{b}, \frac{m^2}{ab-b^2}$, and $\frac{mn}{a^2-b^2}$.
9. $\frac{3}{a+b}, \frac{4}{a-b}$, and $\frac{5}{a^2-b^2}$.
10. $\frac{2-a}{2+a}$ and $\frac{4}{4-a^2}$.
11. $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$.
12. $\frac{x}{x^2-y^2}$ and $\frac{x+y}{x^3-y^3}$.

ADDITION AND SUBTRACTION OF FRACTIONS.

57. I. Add $\frac{a}{m}+\frac{b}{m}$.

$\frac{a}{m}+\frac{b}{m}=\frac{a+b}{m}$. Since the denominators are alike, the sum is a fraction having the same denominator and a numerator equal to the sum of the numerators.

In the same way it can be shown that $\frac{a}{m}-\frac{b}{m}=\frac{a-b}{m}$.

II. Add $\frac{a}{m}+\frac{b}{n}$.

$$\frac{a}{m}+\frac{b}{n}=\frac{an}{mn}+\frac{bm}{mn}=\frac{an+bm}{mn}.$$

When the denominators are not alike, the fractions must first be reduced to equivalent fractions having the lowest common denominator, and then added.

In the same way it can be shown that $\frac{a}{m}-\frac{b}{n}=\frac{an-bm}{mn}$.

To add or subtract fractions, *reduce the fractions, if the denominators are not alike, to equivalent fractions having the lowest common denominator; then add or subtract each numerator, according to the sign which precedes it, and write the result over the lowest common denominator.*

III. Simplify $\frac{a+2b}{b}+\frac{a-2b}{a}-\frac{a^2-4b^2}{2ab}$.

$$\begin{aligned}
&\frac{a+2b}{b}+\frac{a-2b}{a}-\frac{a^2-4b^2}{2ab}\\
&=\frac{2a^2+4ab}{2ab}+\frac{2ab-4b^2}{2ab}-\frac{a^2-4b^2}{2ab}\\
&=\frac{2a^2+4ab+2ab-4b^2-a^2+4b^2}{2ab}\\
&=\frac{a^2+6ab}{2ab}=\frac{a+6b}{2b}
\end{aligned}$$

NOTE 1. Remember to change the sign of every term in the numerator when the fraction is preceded by the sign $-$.

NOTE 2. Every answer should be in its lowest terms.

IV. Simplify $\frac{1}{ac-c^2}-\frac{1}{a^2+ac}$.

$$\begin{aligned}
\frac{1}{ac-c^2}-\frac{1}{a^2+ac}&=\frac{1}{c(a-c)}-\frac{1}{a(a+c)}\\
&=\frac{a^2+ac}{ac(a+c)(a-c)}-\frac{ac-c^2}{ac(a+c)(a-c)}\\
&=\frac{a^2+ac-ac+c^2}{ac(a+c)(a-c)}\\
&=\frac{a^2+c^2}{ac(a^2-c^2)}.
\end{aligned}$$

Note 1. When the denominators are polynomials, write them in the factored form, but perform all multiplications in the numerators.

Note 2. In case the terms of the numerator unite to form 0, the entire answer is 0.

EXAMPLES.

Simplify

1. $\frac{a-4b}{4}-\frac{a-6b}{6}$.

2. $\frac{a-b}{2}+\frac{a+b}{3}-\frac{a}{4}$.

3. $\frac{3x-4}{6}-\frac{x+3}{4}+\frac{4x+9}{9}$.

4. $\frac{1}{m}+\frac{1}{n}+\frac{m-n}{mn}$.

5. $\frac{a-b}{a}+\frac{a+b}{b}-\frac{a^2-b^2}{ab}$.

6. $4x+\frac{3a}{4x^2}+\frac{a+2x}{3x}$.

7. $\frac{m-2n}{2m}+\frac{m-4n}{4m}+\frac{m-8n}{8m}$.

8. $\frac{3x^2-5y^2}{4x^2}+\frac{5x-9y}{8x}-\frac{7}{12}$.

9. $\frac{ab-5}{a^2b^2}-\frac{5b^2-6}{ab^3}+\frac{4a^2-9}{a^3b}$.

10. $\frac{a^2b-3b^2}{5a^2}+\frac{3a^4+3b^4}{5a^2b^2}-\frac{6a^2-ab^2}{10b^2}$.

11. $\frac{1}{a-b}+\frac{1}{a}$.

12. $\frac{x}{y}+\frac{y}{x+y}$.

13. $\frac{1}{m}+\frac{m}{m^2+1}$.

14. $\frac{1}{a-3}+\frac{1}{a-5}$.

15. $\frac{1}{a+b}-\frac{1}{a-b}$.

16. $\frac{1}{x-y}-\frac{x}{(x-y)^2}$.

17. $\frac{a+b}{a-b}+\frac{a-b}{a+b}$.

18. $\frac{a+4}{a+5}-\frac{a-6}{a-5}$.

19. $\dfrac{6}{3x-1}+\dfrac{3x-8}{9x^2-1}$.

20. $\dfrac{m+n}{m-n}-\dfrac{m^2+n^2}{m^2-n^2}$.

21. $\dfrac{x^2+4y^2}{x^2-4y^2}-\dfrac{x-2y}{x+2y}$.

22. $\dfrac{1}{x^2-3x+2}-\dfrac{1}{x^2-x+2}$.

23. $\dfrac{1}{1+a}+\dfrac{1}{1-a}+\dfrac{4}{1-a^2}$.

24. $\dfrac{4}{4-m^2}-\dfrac{1}{2+m}-\dfrac{1}{2-m}$.

25. $\dfrac{3x}{x^2-1}-\dfrac{5}{x+1}+\dfrac{2}{x-1}$.

26. $\dfrac{1}{x+y}+\dfrac{1}{x-y}-\dfrac{2x}{x^2-y^2}$.

27. $\dfrac{a}{a+x}+\dfrac{x}{a-x}-1$.

28. $\dfrac{2}{x+y}+\dfrac{2}{x-y}-\dfrac{1}{x}$.

29. $x+\dfrac{x-z}{x^2+xz}-\dfrac{x+z}{x^2-xz}$.

30. $\dfrac{a}{c}+\dfrac{c}{a+c}+\dfrac{a^2}{a^2+ac}$.

MULTIPLICATION AND DIVISION OF FRACTIONS.

58. I. Multiply $\dfrac{a}{m}$ by $\dfrac{b}{n}$.

$\dfrac{a}{m}\times\dfrac{b}{n}=\dfrac{ab}{mn}$. Let $\dfrac{a}{m}=x$ and $\dfrac{b}{n}=y$. Clearing of fractions, $a=mx$ and $b=ny$. The product of these equations is $ab=mnxy$. Dividing this result by mn, $\dfrac{ab}{mn}=xy$. Multiplying together the original equations, $\dfrac{a}{m}\times\dfrac{b}{n}=xy$. Hence $\dfrac{a}{m}\times\dfrac{b}{n}=\dfrac{ab}{mn}$.

In like manner it can be shown that $\frac{a}{m} \times \frac{b}{n} \times \frac{c}{p} = \frac{abc}{mnp}$, and so on for any number of fractions.

To find the product of several fractions, *multiply the numerators together for a new numerator, and the denominators for a new denominator.*

Note. **Before performing the multiplication every integral expression should be written as a fraction having 1 for its denominator and every mixed expression should be reduced to the fractional form.**

II. Divide $\frac{a}{m}$ by $\frac{b}{n}$.

$\frac{a}{m} \div \frac{b}{n} = \frac{a}{m} \times \frac{n}{b} = \frac{an}{bm}$. Let $\frac{a}{m} \div \frac{b}{n} = x$. Since the dividend is the product of the divisor and the quotient, $\frac{a}{m} = \frac{b}{n} \times x$. Multiplying by $\frac{n}{b}$, $\frac{a}{m} \times \frac{n}{b} = \frac{b}{n} \times x \times \frac{n}{b} = x$. Hence $\frac{a}{m} \div \frac{b}{n} = \frac{a}{m} \times \frac{n}{b} = \frac{an}{bm}$.

To divide a fraction by a fraction, *invert the divisor, and then proceed as in multiplication.*

III. Simplify $\frac{3\,b^2c}{5\,xy} \times \frac{8\,ax}{9\,b^2} \times \frac{3\,y^3}{2\,a^4}$.

$$\frac{\cancel{3}\cancel{b^2}c}{5\cancel{x}\cancel{y}} \times \frac{\overset{4}{\cancel{8}}\cancel{a}\cancel{x}}{\underset{3}{\cancel{9}}\cancel{b^2}} \times \frac{\cancel{3}\overset{y^2}{\cancel{y^3}}}{\cancel{2}\underset{a^3}{\cancel{a^4}}} = \frac{4\,cy^2}{5\,a^3}$$

Every answer should be in its lowest terms. Instead of multiplying the fractions together and then reducing this result to its lowest terms, it is easier to cancel like factors before performing the multiplication.

IV. Simplify $\frac{a^2 - b^2}{c - d} \times \frac{c^2 - d^2}{a^2 + ab} \times \frac{1}{a - b}$.

$$\frac{a^2 - b^2}{c - d} \times \frac{c^2 - d^2}{a^2 + ab} \times \frac{1}{a - b}$$

$$= \frac{\cancel{(a+b)}\cancel{(a-b)}}{\cancel{c-d}} \times \frac{(c+d)\cancel{(c-d)}}{a\cancel{(a+b)}} \times \frac{1}{\cancel{a-b}} = \frac{c+d}{a}.$$

The process of cancellation is made easier by first writing all numerators and denominators in the factored form.

V. Simplify $\frac{a}{b} \times \frac{a^2 - b^2}{a^2 - ab - 2b^2} \div \frac{a - b}{ab + b^2}$.

$$\frac{a}{b} \times \frac{a^2 - b^2}{a^2 - ab - 2b^2} \div \frac{a - b}{ab + b^2}$$

$$= \frac{a}{\cancel{b}} \times \frac{\cancel{(a+b)}\cancel{(a-b)}}{(a-2b)\cancel{(a+b)}} \times \frac{\cancel{b}(a+b)}{\cancel{a-b}} = \frac{a(a+b)}{a-2b}.$$

EXAMPLES.

Simplify

1. $\frac{3a^2x}{4b^2y} \times \frac{7bx}{8ay}$.

2. $\frac{7a^2b^3}{12x^2y} \times \frac{20xy^3}{21a^3b^2}$.

3. $\frac{3a}{7b} \times \frac{7c}{d} \times \frac{bd^2}{b}$.

4. $\frac{m^2}{n^2} \times \frac{p^2}{q^2} \div \frac{m^2}{q^2}$.

5. $\frac{3x}{4y} \times \frac{5y}{6z} \times \frac{7z}{10x}$.

6. $\frac{4a^4}{3b^3} \times \frac{9b^9}{8a^3} \times 2a^2$.

7. $\frac{9a^2c}{30b^2d^4} \times \frac{25b^3d^4}{6a^2c^3} \div 5b$.

8. $\frac{3x^4}{4yz} \times \frac{5y^4}{6x^2z^2} \div \frac{15x^3y^3}{8z^4}$.

9. $\frac{7a^5b^4}{5x^2y^3} \times \frac{4ax}{3by} \div \frac{21a^4b^3}{10x^3y^2}$.

10. $\frac{25m^3n^2}{14x^2y^2} \times \frac{70x^3y}{75nz^2} \div \frac{4m^2n}{3nz}$.

11. $\frac{a+b}{ac} \times \frac{bc}{a}$.

12. $\frac{3a}{a+b} \times \frac{4ab}{a-b}$.

13. $\frac{m-n}{m^2n} \times \frac{mn^2}{m^2-n^2}$.

14. $\frac{14y^3}{3y-12} \times \frac{2y-8}{5y^2}$.

15. $\frac{x^2+5x-24}{4x} \times \frac{5x}{x^2-5x+6}$.

16. $\frac{4x^2-2x}{3y} \div \frac{2x^2-4x}{6y^2}$.

17. $\dfrac{x^2+7x+6}{(x+1)^3} \div \dfrac{4x}{x^3+1}$.

18. $\dfrac{a^2+ab}{a-b} \times \dfrac{(a-b)^2}{ab+b^2}$.

19. $\dfrac{4a^2-b^2}{a^3-4ab^2} \times \dfrac{a^2+2ab}{2a-b}$.

20. $\dfrac{a^2-ax}{ax+x^2} \times \dfrac{a^2-ax-2x^2}{a^2-2ax+x^2}$.

21. $\dfrac{x^2-y^2}{x^2+xy-12y^2} \div \dfrac{x+y}{x^2-9y^2}$.

22. $\dfrac{3a-12}{a^2+3a+2} \times \dfrac{a+1}{a^2-3a-4}$.

23. $\dfrac{m^3-8}{m^2-4} \times \dfrac{m+2}{m^2+2m+4}$.

24. $\dfrac{x^3+y^3}{x^4+x^2y^2+y^4} \times \dfrac{x^3-y^3}{x+y}$.

25. $\dfrac{x+1}{2y} \times \dfrac{x-1}{y+z} \times 3y$.

26. $\dfrac{3ax}{4by} \times \dfrac{a^2-x^2}{b^2-y^2} \times \dfrac{by+y^2}{ax+x^2} \times \dfrac{b-y}{a-x}$.

27. $\dfrac{x^2+x-12}{x^2-16} \times \dfrac{x^2-5x-6}{x^2-5x+6} \div \dfrac{x^3+x}{x^2-2x}$.

28. $\left(b+\dfrac{b}{b-1}\right)\left(1-\dfrac{b}{b+1}\right)$.

29. $\dfrac{a^2-c^2}{a+b} \times \dfrac{a^2-b^2}{ac+c^2} \times \left(1+\dfrac{c}{a-c}\right)$.

30. $\dfrac{2b}{a^2-ab-2b^2} \times \dfrac{a^2-b^2}{4b^2} \times \left(2-\dfrac{a}{a-b}\right)$.

COMPLEX FRACTIONS.

59. A **complex fraction** is one that has a fraction in one or both of its terms. It is a case of division of fractions where the dividend is written above the divisor.

I. Simplify $\dfrac{\dfrac{x}{x-1}-1}{1-\dfrac{1}{x+1}}$.

$$\frac{x}{x-1}-1=\frac{x-x+1}{x-1}=\frac{1}{x-1}.$$

$$1-\frac{1}{x+1}=\frac{x+1-1}{x+1}=\frac{x}{x+1}.$$

$$\frac{1}{x-1}\times\frac{x+1}{x}=\frac{x+1}{x(x-1)}$$

The fraction is treated like an example in division of fractions. The numerator and denominator are simplified separately, and then the numerator is multiplied by the denominator inverted.

In many cases, however, a simpler method is to multiply both terms of the fraction by the lowest common multiple of the denominators of the fractional terms.

II. Simplify $\dfrac{x+\dfrac{1}{x}}{x^2-\dfrac{1}{x^2}}$

$$\frac{x+\dfrac{1}{x}}{x^2-\dfrac{1}{x^2}}=\frac{x^3+x}{x^4-1}=\frac{x(x^2+1)}{(x^2+1)(x^2-1)}=\frac{x}{x^2-1}.$$

Both terms of the fraction are multiplied by x^2, the lowest common multiple of the denominators of the fractional terms.

EXAMPLES.

Simplify

1. $\dfrac{a}{a-\dfrac{b}{c}}$.   2. $\dfrac{1+\dfrac{m}{x}}{\dfrac{n}{x}-1}$.   3. $\dfrac{1+\dfrac{x}{y}}{1+\dfrac{y}{x}}$.   4. $\dfrac{2a+\dfrac{3b}{4}}{3a+\dfrac{5b}{6}}$.

5. $\dfrac{2a - \dfrac{3b}{4c}}{2c - \dfrac{3b}{4a}}.$

7. $\dfrac{a^2 - \dfrac{b^2}{4}}{a^2 - ab + \dfrac{b^2}{4}}.$

9. $\dfrac{\dfrac{a}{b} - \dfrac{c}{d}}{\dfrac{a}{b} + \dfrac{c}{d}}.$

6. $\dfrac{\dfrac{x^2}{y} - y}{\dfrac{x^4}{y^3} - y}.$

8. $\dfrac{x - 7 + \dfrac{6}{x}}{x - 5 + \dfrac{4}{x}}.$

10. $\dfrac{\dfrac{a^2}{b^3} + \dfrac{1}{a}}{\dfrac{a}{b^2} - \dfrac{1}{b} + \dfrac{1}{a}}.$

11. $\dfrac{\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}}{\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{a}}.$

14. $\dfrac{\dfrac{m}{m-1} - \dfrac{m+1}{m}}{\dfrac{m}{m+1} - \dfrac{m-1}{m}}.$

12. $\dfrac{\dfrac{m}{n^2} + \dfrac{n}{m^2}}{\dfrac{1}{m^2} + \dfrac{1}{mn} + \dfrac{1}{n^2}}.$

15. $\dfrac{\dfrac{a+b}{a-b} - \dfrac{a-b}{a+b}}{\dfrac{a+b}{a-b} + \dfrac{a-b}{a+b}}.$

13. $\dfrac{a + \dfrac{ax}{a-x}}{a - \dfrac{ax}{a+x}}.$

16. $\dfrac{\dfrac{a+b}{x+y} + \dfrac{a-b}{x-y}}{\dfrac{a+b}{x-y} + \dfrac{a-b}{x+y}}.$

CHAPTER XV.

SPECIAL PROBLEMS.

50. In Chapter III. we studied fractional equations in which all the denominators were monomials. We are now prepared to study fractional equations which contain polynomial denominators.

I. Solve $\frac{1}{x-1}+\frac{2}{x-2}=\frac{3}{x-3}$.

$$\frac{1}{x-1}+\frac{2}{x-2}=\frac{3}{x-3}.$$

Multiplying both members by $(x-1)(x-2)(x-3)$, the L.C.D.,
$(x-2)(x-3)+2(x-1)(x-3)=3(x-1)(x-2)$.

Performing the multiplications indicated,

$$x^2-5x+6+2x^2-8x+6=3x^2+9x+6.$$
$$x^2+2x^2-3x^2-5x-8x+9x=6-6-6.$$
$$-4x=-6.$$
$$x=1\tfrac{1}{2}.$$

EXAMPLES.

Solve the following equations:

1. $\frac{x-2}{x+4}=\frac{2}{5}$.

2. $\frac{x-3}{x+3}+\frac{3}{x}=1$.

3. $\frac{(x+6)^2}{x-3}=\frac{2x+3}{2}$.

4. $\frac{3x^2+4x}{3x+1}+\frac{1}{4x}=x+1$.

5. $\frac{(3x-4)(2x+1)}{6x(x-1)}-1=0$

6. $\frac{3x+4}{x}+\frac{2x-3}{x+1}=5$.

7. $\frac{2}{1-3x}-\frac{3}{1-2x}=0$.

8. $\frac{3x+5}{4x+2} = \frac{3x-5}{4x-8}$.

9. $\frac{5x+8}{2x+5} = \frac{5x-12}{2x-5}$.

10. $\frac{3x-2}{x(x-2)} = \frac{3x-5}{x^2-4}$.

11. $\frac{x+6}{2(x-2)} = \frac{x+16}{4(x-2)} + \frac{1}{3}$.

12. $\frac{2}{x+1} - \frac{1}{x} = \frac{1}{x^2+x}$.

13. $\frac{3}{x-1} + \frac{2}{x+1} = \frac{26}{x^2-1}$.

14. $\frac{x+1}{x-1} - \frac{7}{x+1} = \frac{x^2}{x^2-1}$.

15. $\frac{8}{x-3} + \frac{3}{x+4} = \frac{45}{x^2+x-12}$.

LITERAL EQUATIONS.

51. **Literal equations** are equations in which some or all of the unknown quantities are represented by letters.

I. Solve $\frac{1}{m} + \frac{m}{m+x} = \frac{m+x}{mx}$.

$$\frac{1}{m} + \frac{m}{m+x} = \frac{m+x}{mx}.$$

Clearing of fractions, $x(m+x) + m^2x = (m+x)(m+x)$.

Removing parentheses, $mx + x^2 + m^2x = m^2 + 2mx + x^2$.

$$x^2 - x^2 + mx + m^2x - 2mx = m^2.$$

$$m^2x - mx = m^2.$$

$$(m^2 - m)x = m^2.$$

$$x = \frac{m^2}{m^2-m} = \frac{m}{m-1}.$$

EXAMPLES.

Solve the following equations:

1. $2a - bx = 5c - 3bx.$

2. $a(x - c) + ax = cx - a.$

3. $(x + a)(x - a) = x^2 - 2ax + 2a^2.$

4. $(a - 2b)x - (a + 2b)x = 4b^2.$

5. $(x + m)(x - n) = (x + m - n)^2.$

6. $\frac{x}{m} + \frac{x}{n} = 1.$

7. $\frac{ac}{x} = ab + c - \frac{1}{x}.$

8. $\frac{ax - c}{b} + a = \frac{x + ab}{b}.$

9. $a^2x - \frac{2b - 1}{x} = \frac{a^2(x^2 - 1)}{x}.$

10. $\frac{m(p^2 + x^2)}{px} = mq + \frac{mx}{p}.$

11. $\frac{m}{nx} + \frac{n}{mx} = m^2 - n^2.$

12. $\frac{x^2 - a}{bx} + \frac{x + a}{b} = \frac{2x}{b} + \frac{a}{x}.$

13. $\frac{4ax - 3b}{4b} - \frac{ax - a}{3b} = \frac{ax}{b} - \frac{3}{4}.$

14. $\frac{x + 3}{x - 3} = \frac{a + b}{a - b}.$

15. $\frac{2a + x}{b + x} + \frac{x}{b - x} = \frac{a^2 + b^2}{b^2 - x^2}.$

WORK PROBLEMS.

52. Problems concerning work should be solved by considering the fractional part of the work that can be done in a unit of time. For example, if a man can do a piece of work in 6 days, he can do $\frac{1}{6}$ of the work in one day; if he can do the work in x days, the part he can do in one day is denoted by $\frac{1}{x}$.

I. A can do a piece of work in 6 days, and B can do it in 8 days. How long will it take both together to do it?

$x =$ the number of days it will take both together.

$\frac{1}{6} =$ the part A can do in one day.

$\frac{1}{8} =$ the part B can do in one day.

$\frac{1}{x} =$ the part both together can do in one day.

$$\frac{1}{6}+\frac{1}{8}=\frac{1}{x}.$$
$$4x+3x=24.$$
$$7x=24.$$
$$x=3\tfrac{3}{7}.$$

Ans. $3\frac{3}{7}$ days.

EXAMPLES.

1. A can do a piece of work in 3 days and B can do it in 5 days. How long will it take both together to do it?

2. A cistern can be filled by a pipe in $3\frac{1}{2}$ hours, and by another in $2\frac{1}{3}$ hours; how long will it take both together to fill it?

3. A can do a piece of work in 4 days, B in 6 days, and C in 9 days. How long will it take them to do it, working together?

4. A and B can do a piece of work in 6 days. A, working alone, can do the same work in 10 days. How long would it take B to do it?

5. A can do a piece of work in 8 days and B can do it in 12 days. A, B, and C, working together, can do it in 4 days. How long will it take C, working alone, to do the work?

6. A cistern can be filled by one pipe in 5 hours and can be emptied by another pipe in 4 hours. In what time will the cistern be filled if both pipes are open?

7. A cistern has two pipes, one of which can fill it in 4 hours, and the other in 6 hours; a third pipe can empty it in 3 hours. If all three pipes are open, how long will it take to fill the cistern?

8. One pipe can fill a cistern in 5 hours, and a second pipe can fill it in 6 hours. The first is kept open 3 hours, and then the second is opened. How long will it take to finish filling the cistern?

9. A can do a piece of work in 16 days; B can work twice as fast as A, and C twice as fast as B. How long will it take them all together to do it?

10. A can do a piece of work in 12 days, A and B can do it in 4 days, and A and C can do it in 6 days. In how many days can B and C together do it?

11. A can do a piece of work in 6 days, working 8 hours a day. B can do the same work in 8 days, working 9 hours a day. They decide to work together, and to finish the work in 4 days. How many hours a day must they work?

CLOCK PROBLEMS.

53. The solution of clock problems depends upon the fact that the minute-hand moves twelve times as fast as the hour-hand.

I. At what time between 4 and 5 o'clock are the hands of a clock together?

$x =$ the number of minutes past 4 o'clock.

That is, $x =$ the number of minute-spaces the minute-hand moves.

$x - 20 =$ the number of minute-spaces the hour-hand moves.

Since the minute-hand moves twelve times as fast as the hour-hand,

$$x = 12(x - 20).$$
$$x = 12x - 240.$$
$$x - 12x = -240.$$
$$-11x = -240.$$
$$x = 21\tfrac{9}{11}.$$

Ans. $21\frac{9}{11}$ minutes past 4 o'clock.

II. At what time between 3 and 4 o'clock are the hands of a clock opposite each other?

$x =$ the number of minutes past 3 o'clock.

That is, $x =$ the number of minute-spaces the minute-hand moves.

$x - 15 - 30 =$ the number of minute-spaces the hour-hand moves.

$$x = 12(x - 15 - 30).$$
$$x = 12(x - 45).$$
$$x = 12x - 540.$$
$$x - 12x = -540.$$
$$-11x = -540.$$
$$x = 48\tfrac{4}{11}.$$

Ans. $48\frac{4}{11}$ minutes past 3 o'clock.

EXAMPLES.

1. At what time between 2 and 3 o'clock are the hands of a clock together?

2. At what time between 8 and 9 o'clock are the hands of a clock together?

3. At what time between 1 and 2 o'clock are the hands of a clock opposite each other?

4. At what time between 9 and 10 o'clock are the hands of a clock opposite each other?

5. At what time between 5 and 6 o'clock are the hands of a clock at right angles to each other?

6. At what time between 10 and 11 o'clock are the hands of a clock at right angles to each other?

7. At what time between 3 and 4 o'clock is the minute-hand of a clock five minutes in advance of the hour-hand?

8. At what time between 7 and 8 o'clock is the hour-hand of a clock ten minutes in advance of the minute-hand?

DISTANCE PROBLEMS.

54. The principle used in the solution of distance problems can be shown by a simple illustration. If a man walks 3 miles an hour, he will walk 24 miles in 8 hours. Using d, r, and t to denote respectively distance, rate, and time, $d = rt$. The formula is often used in the form $\frac{d}{r} = t$.

I. A man sets out and walks at the rate of 4 miles an hour; 6 hours later a boy starts in pursuit, riding on a bicycle at the rate of 12 miles an hour. In how many hours will the boy overtake the man?

$x =$ the number of hours the boy rides.

$x + 6 =$ the number of hours the man walks.

Since $d = rt$, $12x =$ the number of miles the boy rides.

$4(x + 6) =$ the number of miles the man walks.

$$12x = 4(x + 6).$$
$$12x = 4x + 24.$$
$$12x - 4x = 24.$$
$$8x = 24.$$
$$x = 3.$$

Ans. 3 hours.

II. A man walks into the country at the rate of 3 miles an hour, and returns by electric cars running at the rate of 9 miles an hour. His entire trip lasts 8 hours. How far does he walk?

$x =$ the number of miles he walks.

$\frac{x}{3} =$ the number of hours he walks.

$\frac{x}{9} =$ the number of hours he rides in the cars.

$$\frac{x}{3} + \frac{x}{9} = 8.$$
$$3x + x = 72.$$
$$4x = 72.$$
$$x = 18.$$

Ans. 18 miles.

EXAMPLES.

1. A train running at the rate of 30 miles an hour has been gone 2 hours, when a second train follows at the rate of 42 miles an hour. In how many hours will the second train overtake the first?

2. Two steamers on the same line start two days apart. The first runs 280 miles a day, and the second 350 miles a day. In how many days will the second overtake the first?

3. Two men start from two cities 320 miles apart and walk toward each other. The first walks 22 miles a day, and the second 18 miles a day. In how many days will they meet?

4. The distance from Boston to Springfield is 99 miles. A man starts from Boston and rides at the rate of 8 miles an hour; at the same time another man starts from Springfield and walks at the rate of 3 miles an hour. How far from Boston will they meet?

5. A man who walks at the rate of 4 miles an hour starts 2 hours after another man, and overtakes him in 6 hours. Find the rate of the slower man.

6. A train can run 200 miles in a certain time. If the rate is increased 5 miles an hour, it can run 25 miles more in the same time. Find the rate of the train.

7. In going the entire length of a railroad, a freight train running 15 miles an hour requires 8 hours more time than a passenger train running 35 miles an hour. Find the length of the railroad.

8. A man has just $3\frac{1}{2}$ hours at his disposal; how far may he ride in a car which runs at the rate of 10 miles an hour so as to return home in time, walking back at the rate of 4 miles an hour?

9. A and B start from the same place walking at different rates. When A has walked 12 miles, B doubles his pace, and $4\frac{1}{2}$ hours later overtakes A. If A's rate is 4 miles an hour, what is B's rate at first?

10. The distance from M to N is 120 miles. A train leaving M at a certain rate meets with an accident after running 90 miles; its rate is then reduced one-half, and it reaches N one hour late. Find the original rate.

CHAPTER XVI.

SQUARE AND CUBE ROOTS.

SQUARE ROOT.

55. In § 30 we learned that the square of $a+b$ is $a^2+2ab+b^2$. Hence the square root of $a^2+2ab+b^2$ is $a+b$. Thus we see that the first term, a, of the root is the square root of the first term, a^2, of the given expression.

Subtracting a^2 from the given expression, the remainder is $2ab+b^2$, or $(2a+b)b$. Hence if we divide $2ab+b^2$ by $2a+b$, the quotient, b, is the second term of the root.

The divisor $2a+b$ is known only in part, so we take $2a$ as a *trial divisor*; dividing $2ab$ by $2a$, the quotient is b. Hence b is the second term of the root, and the *complete divisor* is $2ab+b^2$.

Multiplying $2a+b$ by b, we obtain $2ab+b^2$, and there is no remainder.

The following is a convenient arrangement of the work:

$$\begin{array}{r|l}
 & a^2+2ab+b^2\,(a+b \\
 & \underline{a^2} \\
2a+b\,) & 2ab+b^2 \\
 & \underline{2ab+b^2}
\end{array}$$

I. Find the square root of $9x^2-24xy+16y^2$.

$$\begin{array}{r|l}
 & 9x^2-24xy+16y^2\,(3x-4y \\
 & \underline{9x^2} \\
6x-4y\,) & -24xy+16y^2 \\
 & \underline{-24xy+16y^2}
\end{array}$$

The square root of $9x^2$ is $3x$, which is the first term of the root. The trial divisor is two times $3x$, or $6x$. Dividing $-24xy$ by $6x$, the second term of the root is $-4y$, and the complete divisor is $6x-4y$. Multiplying $6x-4y$ by $-4y$, we obtain $-24xy+16y^2$, and there is no remainder. Hence the square root of $9x^2-24xy+16y^2$ is $3x-4y$.

The same method can be applied to longer expressions. At each stage of the work the trial divisor is twice the root already found.

II. Find the square root of $16m^4 - 32m^3 + 17m^2 - 6m + 1$.

$$
\begin{array}{r|l}
 & 16m^4 - 32m^3 + 17m^2 - 6m + 1\,(4m^2 - 3m + 1 \\
 & 16m^4 \\
\hline
8m^2 - 3m & -32m^3 + 17m^2 - 6m + 1 \\
 & -32m^3 + 9m^2 \\
\hline
8m^2 - 6m + 1 & 8m^2 - 6m + 1 \\
 & 8m^2 - 6m + 1 \\
\hline
\end{array}
$$

EXAMPLES.

Find the square root of

1. $4m^2 + 20mn^2 + 25n^4$.
2. $9x^4 - 42x^2y^2 + 49y^4$.
3. $16y^6 + 72y^3z^3 + 81z^6$.
4. $25p^2 - 80pq + 64q^2$.
5. $x^8 + 22x^4 + 121$.
6. $4c^{10} - 60c^5d^2 + 225d^4$.
7. $36a^6 + 156a^3c^3 + 169c^6$.
8. $196b^{12} - 28b^6 + 1$.
9. $a^4 - 2a^3b + 3a^2b^2 - 2ab^3 + b^4$.
10. $4b^4 + 4b^3 - 7b^2 - 4b + 4$.
11. $1 - 6c + 13c^2 - 12c^3 + 4c^4$.
12. $m^4 - 4m^3n + 8mn^3 + 4n^4$.
13. $9x^4 - 24x^3 - 14x^2 + 40x + 25$.
14. $a^2 - 2ab + 2ac + b^2 - 2bc + c^2$.
15. $x^4 - 6x^2y + 4x^2 + 9y^2 - 12y + 4$.
16. $a^6 - 2a^5 + 3a^4 - 4a^3 - 3a^2 - 2a + 1$.
17. $x^6 - 4x^5 + 6x^3 + 8x^2 + 4x + 1$.
18. $a^6 - 4a^5 - 2a^4 + 16a^3 + a^2 - 12a + 4$.

56. The first step in finding the square root of a number is to determine the number of figures in the root. $1^2 = 1$, $10^2 = 100$, $100^2 = 10000$, $1000^2 = 1000000$, and so on. Hence the square of any number between 1 and 10 is a number between 1 and 100, the square of any number between 10 and 100 is a number between 100 and 10000, the square of any number between 1000 and 10000 is a number between 10000 and 1000000, and so on. Thus we see that the square of a number contains twice as many figures as the number itself, or twice as many less one. If, therefore, a number be separated into periods of two figures each by placing a dot over every alternate figure, beginning with the units' figure, the number of figures in the root equals the number of periods.

NOTE 1. The left-hand period has but one figure when the number consists of an odd number of figures.

NOTE 2. The principle applies also to decimals, because the square of a decimal contains twice as many decimal places as the decimal itself.

I. Find the square root of 4096.

$$\begin{array}{r|l} & \dot{4}0\dot{9}6\ (60+4=64 \\ & \underline{3600} \\ 120+4=124) & 496 \\ & \underline{496} \end{array}$$

Or more briefly

$$\begin{array}{r|l} & \dot{4}0\dot{9}6\ (64 \\ & \underline{36} \\ 124) & 496 \\ & \underline{496} \end{array}$$

Since the number consists of two periods, the square root will consist of two figures. The square of the number denoted by the tens' digit of the root must be the largest square in 40 hundreds; that is, 36 hundreds. Hence the tens' figure of the root is 6, and the number 60 corresponds to a in the algebraic process of the preceding section. Subtracting 3600, the square of 60, the remainder is 496. The trial divisor is 120; dividing 496 by 120, we obtain 4 as the units' figure of the root, which corresponds

to b of the algebraic process. The complete divisor is $120 + 4$, or 124. Multiplying 124 by 4, we obtain 496, and there is no remainder. Hence the square root of 4096 is 64.

In the shorter arrangement of work, 12 may be considered as the trial divisor, and the units' figure is obtained by dividing 49 by 12, which gives the same result as dividing 496 by 120.

II. Find the square root of 843.3216.

$$\begin{array}{rl}
 & \dot{8}4\dot{3}.\dot{3}2\dot{1}6\,(29.04 \\
 & 4 \\ \hline
49\,) & 443 \\
 & 441 \\ \hline
5804\,) & 23216 \\
 & 23216 \\ \hline
\end{array}$$

After finding 29 as the first two figures of the root, the trial divisor is 58; 58 is not contained once in 23, so the next figure of the root is 0. Then the next trial divisor is 580, which can be used at once by bringing down another period. Since there are two periods of whole numbers, the root has two figures before the decimal point.

The square root of a fraction in its lowest terms may be obtained by taking the square root of both numerator and denominator when they are perfect squares. For example, the square root of $\frac{36}{49}$ is $\frac{6}{7}$; the square root of $11\frac{1}{9}$ is the same as the square root of $\frac{100}{9}$, which equals $\frac{10}{3}$, or $3\frac{1}{3}$. When either numerator or denominator is not a perfect square, reduce the fraction to a decimal, and then find the square root.

III. Find the square root of $\frac{3}{8}$.

$\frac{3}{8} = 0.375$

$$\begin{array}{rl}
 & \dot{0}.\dot{3}75\dot{0}\,(0.61237 \\
 & 36 \\ \hline
121\,) & 150 \\
 & 121 \\ \hline
1222\,) & 2900 \\
 & 2444 \\ \hline
12243\,) & 45600 \\
 & 36729 \\ \hline
122467\,) & 887100
\end{array}$$

A zero must be annexed to complete the second period after the decimal point. Other periods of two zeros each may be brought down as they are needed. When the square root of a number cannot be determined exactly, the approximation should be carried to as many decimal places as is demanded.

EXAMPLES.

Find the square root (to four decimal places when the number is not a perfect square) of

1.	729.	**7.**	82.2649.	**13.**	$\frac{49}{169}$.
2.	1681.	**8.**	446.0544.	**14.**	$\frac{81}{441}$.
3.	6241.	**9.**	642722.89.	**15.**	$\frac{1}{3}$.
4.	654481.	**10.**	0.4.	**16.**	$\frac{3}{7}$.
5.	4884100.	**11.**	0.225.	**17.**	$30\frac{1}{4}$.
6.	7106.49.	**12.**	62.5.	**18.**	$6\frac{2}{3}$.

CUBE ROOT.

57. By performing the multiplication we learn that the cube of $a+b$ is $a^3+3a^2b+3ab^2+b^3$. Hence the cube root of $a^3+3a^2b+3ab^2+b^3$ is $a+b$. Thus we see that the first term, a, of the root is the cube root of the first term, a^3, of the given expression.

Subtracting a^3 from the given expression, the remainder is $3a^2b+3ab^2+b^3$, or $(3a^2+3ab+b^2)b$. Hence, if we divide $3a^2b+3ab^2+b^3$ by $3a^2+3ab+b^2$, the quotient, b, is the second term of the root.

The divisor $3a^2+3ab+b^2$ is known only in part, so we take $3a^2$ as a *trial divisor*; dividing $3a^2b$ by $3a^2$, the quotient is b. Hence b is the second term of the root, and the *complete divisor* is $3a^2+3ab+b^2$.

Multiplying $3a^2+3ab+b^2$ by b, we obtain $3a^2b+3ab^2+b^3$, and there is no remainder.

The following is a convenient arrangement of the work:

$$\begin{array}{r|l}
 & a^3+3a^2b+3ab^2+b^3\,(a+b \\
 & \underline{a^3\qquad\qquad\qquad\qquad} \\
3a^2+3ab+b^2\,) & 3a^2b+3ab^2+b^3 \\
 & \underline{3a^2b+3ab^2+b^3}
\end{array}$$

I. Find the cube root of $27x^3 - 108\,x^2y + 144\,xy^2 - 64\,y^3$.

$$
\begin{array}{r|l}
 & 27\,x^3 - 108\,x^2y + 144\,xy^2 - 64\,y^3\,(3\,x - 4y \\
 & \underline{27\,x^3} \\
27\,x^2 - 36\,xy + 16\,y^2\,) & -108\,x^2y + 144\,xy^2 - 64\,y^3 \\
 & \underline{-108\,x^2y + 144\,xy^2 - 64\,y^3}
\end{array}
$$

The cube root of $27\,x^3$ is $3\,x$, which is the first term of the root. The trial divisor is three times $(3\,x)^2$, or $27\,x^2$. Dividing $-108\,x^2y$ by $27\,x^2$, the second term of the root is $-4\,y$. The complete divisor is determined by adding to the trial divisor three times the product of $3\,x$ and $-4\,y$ and the square of $-4\,y$; thus the complete divisor is $27\,x^2 - 36\,xy + 16\,y^2$. Multiplying the complete divisor by $-4\,y$, we obtain $-108\,x^2y + 144\,xy^2 - 64\,y^3$, and there is no remainder. Hence the cube root of $27\,x^3 - 108\,x^2y + 144\,xy^2 - 64\,y^3$ is $3\,x - 4\,y$.

The same method can be applied to longer expressions. At each stage of the work the trial divisor is three times the square of the root already found.

II. Find the cube root of $a^6 + 6\,a^5 - 40\,a^3 + 96\,a - 64$.

$$
\begin{array}{r|l}
 & a^6 + 6\,a^5 - 40\,a^3 + 96\,a - 64\,(a^2 + 2\,a - 4 \\
 & \underline{a^6} \\
3\,a^4 + 6\,a^3 + 4\,a^2\,) & 6\,a^5 - 40\,a^3 + 96\,a - 64 \\
 & \underline{6\,a^5 + 12\,a^4 + 8\,a^3} \\
3\,a^4 + 12\,a^3 + 12\,a^2 & -12\,a^4 - 48\,a^3 + 96\,a - 64 \\
-12\,a^2 - 24\,a & \\
\underline{16} & \\
\underline{3\,a^4 + 12\,a^3 \qquad - 24\,a + 16} & \underline{-12\,a^4 - 48\,a^3 + 96\,a - 64}
\end{array}
$$

The cube root of a^6 is a^2, which is the first term of the root. The trial divisor is three times $(a^2)^2$, or $3\,a^4$. Dividing $6\,a^5$ by $3\,a^4$, the second term of the root is $2\,a$. The complete divisor is $3\,a^4 + 3\,a^2 . 2\,a + (2\,a)^2$, or $3\,a^4 + 6\,a^3 + 4\,a^2$.

After finding two terms of the root, the second trial divisor is $3\,(a^2 + 2\,a)^2$, or $3\,a^4 + 12\,a^3 + 12\,a^2$. Dividing $-12\,a^4$ by $3\,a^4$, the third term of the root is -4. In order to obtain the second complete divisor, we must add to the trial divisor $3\,(a^2 + 2\,a)\,(-4)$, or $-12\,a^2 - 24\,a$, and $(-4)^2$, or 16. Hence the second complete divisor is $3\,a^4 + 12\,a^3 - 24\,a + 16$.

EXAMPLES.

Find the cube root of

1. $x^3 - 9x^2 + 27x - 27.$
2. $y^3 - 12y^2z + 48yz^2 - 64z^3.$
3. $m^6 + 6m^4 + 12m^2 + 8.$
4. $8x^3 - 12x^2 + 6x - 1.$
5. $27c^3 + 54c^2d + 36cd^2 + 8d^3.$
6. $a^6 - 3a^5 + 6a^4 - 7a^3 + 6a^2 - 3a + 1.$
7. $x^6 - 9x^5 + 21x^4 + 9x^3 - 42x^2 - 36x - 8.$
8. $8a^6 + 12a^5 - 30a^4 - 35a^3 + 45a^2 + 27a - 27.$
9. $27y^6 - 54y^5 + 90y^4 - 80y^3 + 60y^2 - 24y + 8.$
10. $64a^6 - 96a^5b + 40a^3b^3 - 6ab^5 - b^6.$

58. In finding the cube root of a number, the first step is to divide the number into periods. Since $1^3 = 1$, $10^3 = 1000$, $100^3 = 1000000$, and so on, the cube of any number between 1 and 10 is a number between 1 and 1000, the cube of any number between 10 and 100 is a number between 1000 and 1000000, and so on. Thus we see that the cube of a number contains three times as many figures as the number itself, or three times as many less one or two. If, therefore, a number be separated into periods of three figures each by placing a dot over every third figure, beginning with the units' figure, the number of figures in the root equals the number of periods.

NOTE 1. The left-hand period may contain but one or two figures.

NOTE 2. The principle applies also to decimals, because the cube of a decimal contains three times as many decimal places as the decimal itself.

I. Find the cube root of 103823.

$$\begin{array}{rr|l} & & 10\dot{3}82\dot{3}\,(47 \\ & & 64 \\ \hline 3\times 40^2 = & 4800 & 39823 \\ 3\times 40\times 7 = & 840 & \\ 7^2 = & 49 & \\ \hline & 5689 & 39823 \\ \hline \end{array}$$

Since the number consists of two periods, the cube root will consist of two figures. The cube of the number denoted by the tens' digit of the root must be the largest cube in 103 thousands; that is, 64 thousands. Hence the tens' figure of the root is 4, and the number 40 corresponds to a in the algebraic process of the preceding section. Subtracting 64000, the cube of 40, the remainder is 39823. The trial divisor is 3×40^2, or 4800; dividing 39823 by 4800, we obtain 7 as the units' figure of the root, which corresponds to b of the algebraic process. The complete divisor is $3\times 40^2 + 3\times 40\times 7 + 7^2$, or 5689. Multiplying 5689 by 7, we obtain 39823, and there is no remainder. Hence the cube root of 103823 is 47.

II. Find the cube root of 1906.624.

$$\begin{array}{rr|l} & & \dot{1}90\dot{6}.62\dot{4}\,(12.4 \\ & & 1 \\ \hline 3\times 10^2 = & 300 & 906 \\ 3\times 10\times 2 = & 60 & \\ 2^2 = & 4 & \\ \hline & 364 & 728 \\ \hline 3\times 120^2 = & 43200 & 178624 \\ 3\times 120\times 4 = & 1440 & \\ 4^2 = & 16 & \\ \hline & 44656 & 178624 \\ \hline \end{array}$$

The cube root of a fraction in its lowest terms may be obtained by taking the cube root of both numerator and denominator when they are perfect cubes. When either numerator or denominator is not a perfect cube, reduce the fraction to a decimal, and then find the cube root.

III. Find the cube root of $\frac{7}{20}$ to four decimal places.

$$
\begin{array}{rrl|l}
\frac{7}{20} = .35 & & & .35\dot{0}\ (.7047 \\
 & & & 343 \\
 & 3 \times 700^2 = & 1470000 & 7000000 \\
 & 3 \times 700 \times 4 = & 8400 & \\
 & 4^2 = & 16 & \\
\hline
 & & 1478416 & 5913664 \\
\hline
 & & & 1086336
\end{array}
$$

The first trial divisor is 3×70^2, or 14700. Since 14700 is not contained once in 7000, the next figure of the root is 0. Then the trial divisor is 3×700^2, or 1470000, which can be used at once by bringing down another period of three zeros.

Since the first two figures of the next trial divisor are the same as those of the preceding divisor, the last figure of the root may be found by simple division. 10863360 divided by 1478416 gives 7 as the last figure of the root.

EXAMPLES.

Find the cube root (to three decimal places when the number is not a perfect cube) of

1.	13824.	7.	16.581375.	13.	$\frac{125}{216}$.
2.	97336.	8.	101847.563.	14.	$\frac{3}{4}$.
3.	148877.	9.	354894912.	15.	$\frac{5}{11}$.
4.	571.787.	10.	0.27.	16.	$15\frac{5}{8}$.
5.	912673000.	11.	0.075.	17.	$22\frac{1}{2}$.
6.	4019.679.	12.	1.025.	18.	$33\frac{1}{3}$.

CHAPTER XVII.

QUADRATIC EQUATIONS.

59. An equation which contains the square of the unknown quantity, but no higher power, is called a **quadratic equation**, or an **equation of the second degree.**

If the equation contains both the square and the first power of the unknown quantity, it is called a **complete** or **affected quadratic equation.** If the equation contains only the square of the unknown quantity, it is called an **incomplete** or **pure quadratic equation.** For example, $3x^2 + 2x = 16$ is a complete quadratic equation, and $7x^2 = 28$ is an incomplete quadratic equation.

INCOMPLETE QUADRATIC EQUATIONS.

60. I. Solve $\dfrac{5x^2 - 8}{3} + \dfrac{7 - 2x^2}{4} = 3\frac{3}{4}$.

Clearing of fractions, $20x^2 - 32 + 21 - 6x^2 = 45.$

$$20x^2 - 6x^2 = 45 + 32 - 21.$$
$$14x^2 = 56.$$
$$x^2 = 4.$$

Extracting the square root, $x = \pm 2.$

Since $(+2)^2 = 4$ and $(-2)^2 = 4$, the square root of 4 is either $+2$ or -2. This is indicated by the double sign $\pm$ (read *plus or minus*).

II. Solve $\dfrac{3}{x^2 + 2} = \dfrac{2}{x^2 - 1}$.

Clearing of fractions, $3x^2 - 3 = 2x^2 + 4.$

$$3x^2 - 2x^2 = 4 + 3.$$
$$x^2 = 7.$$
$$x = \pm\sqrt{7}$$

Note. When the value of x^2 is not a perfect square, the value of x cannot be obtained exactly. It is customary to use the radical sign in expressing the value of x. By extracting the square root of 7 the value of x can be found to as many decimal places as is desirable.

EXAMPLES.

Solve the following equations:

1. $3x^2 - 5 = 2x^2 + 4.$

2. $4x^2 - 18 = x^2 + 12.$

3. $x^2 + 1 = \frac{x^2}{3} + 3.$

4. $\frac{x+4}{3} = \frac{3}{x-4}.$

5. $\frac{x^2-6}{5} = \frac{x^2+6}{7}.$

6. $\frac{x}{2} + \frac{2}{x} = \frac{x}{8} + \frac{8}{x}.$

7. $\frac{x-5}{5x-1} = \frac{5x-1}{x-5}.$

8. $(x-5)(x+4) + (x+5)(x-4) = 10.$

9. $\frac{11x^2}{2} - 12x + 60 = (3x-2)^2.$

10. $10 - \frac{x+25}{x^2} = 2 - \frac{x-25}{x^2}.$

11. $\frac{3x^2-4}{2} - \frac{3x^2-4}{4} - \frac{5x^2-3}{7} = 0.$

12. $\frac{x+1}{x-1} + \frac{x-1}{x+1} = \frac{13}{6}.$

COMPLETE QUADRATIC EQUATIONS.

61. Since $x^2 + 2ax + a^2$ is the square of $x + a$, the binomial $x^2 + 2ax$ will become a perfect square if a^2 is added. It should be observed that this added term is the square of one-half the coefficient of x. The process of adding this term is called *completing the square.* For example, if we have the binomial $x^2 + 10x$, we obtain the perfect square $x^2 + 10x + 25$ by adding the square of one-half of 10.

The solution of a complete quadratic equation is based on this principle of completing the square.

I. Solve $x^2 + 8x = 33$.

Adding the square of one-half of 8 to both members,

$$x^2 + 8x + 16 = 33 + 16 = 49.$$

Extracting the square root, $x + 4 = \pm 7$.

Transposing, $$x = -4 + 7 \text{ or } -4 - 7.$$

$$x = 3 \text{ or } -11.$$

II. Solve $5x^2 - 4x = 12$.

Dividing by 5, $$x^2 - \frac{4}{5}x = \frac{12}{5}.$$

Completing the square, $$x^2 - \frac{4}{5}x + \left(\frac{2}{5}\right)^2 = \frac{12}{5} + \frac{4}{25} = \frac{64}{25}.$$

$$x - \frac{2}{5} = \pm\frac{8}{5}.$$

$$x = \frac{2}{5} + \frac{8}{5} \text{ or } \frac{2}{5} - \frac{8}{5}.$$

$$x = 2 \text{ or } -\frac{6}{5}.$$

NOTE 1. In extracting the square root, notice that the sign joining the two terms of the binomial is the same as the sign of the second term of the perfect square.

NOTE 2. When the added term is a fraction, it is advisable to write it as a square in the first member and in the expanded form in the second member.

III. Solve $-6x^2 + 13x = 6.$

Dividing by -6, $$x^2 - \frac{13}{6}x = -1.$$

Completing the square, $$x^2 - \frac{13}{6}x + \left(\frac{13}{12}\right)^2 = -1 + \frac{169}{144} = \frac{25}{144}.$$

$$x - \frac{13}{12} = \pm\frac{5}{12}.$$

$$x = \frac{13}{12} + \frac{5}{12} \text{ or } \frac{13}{12} - \frac{5}{12}.$$

$$x = \frac{3}{2} \text{ or } \frac{2}{3}.$$

Note. In finding one-half of a fraction, divide the numerator by 2 if it is an even number; otherwise, multiply the denominator by 2.

If an equation contains fractions or parentheses, it should be reduced to the form $x^2 + bx = c$ before completing the square.

EXAMPLES.

Solve the following equations:

1. $x^2 + 8x = 33.$	**7.** $7x^2 + 4x = 75.$
2. $x^2 - 9x = 10.$	**8.** $12x^2 - 13x = 90.$
3. $x^2 + x = 20.$	**9.** $8x^2 + 6 = 19x.$
4. $x^2 - 10x = 24.$	**10.** $12 + 17x - 5x^2 = 0.$
5. $3x^2 - 5x = 50.$	**11.** $x + 1 = 12x^2.$
6. $6x^2 - x = 1.$	**12.** $16x - 15 = 4x^2.$

13. $(x+2)(3x+4) = 6x^2 + 17.$

14. $(4x-7)(3x-7) = x^2 - 2x + 7.$

15. $4(x+2) = 3(x-2)(x-5).$

16. $(x-5)^2 = 25(x+1)^2.$

17. $\frac{2x}{3}+\frac{3}{2x}=2\frac{1}{2}.$

18. $\frac{x-4}{6}-\frac{2x-6}{5}=\frac{x-2}{2x}.$

19. $\frac{25}{x}-\frac{65-3x}{2x^2}=4.$

20. $x-\frac{1}{x}=1\frac{1}{2}-\frac{x}{2}+\frac{2}{x}.$

21. $\frac{1}{1+x}+\frac{1+x}{x}=\frac{5}{2}.$

22. $\frac{x+3}{x-1}-\frac{5-x}{2x}=2\frac{2}{3}.$

23. $\frac{x+2}{x+4}=\frac{x-2}{2x-7}.$

24. $\frac{2}{5-x}+\frac{1}{2x-5}=2.$

25. $\frac{1}{2(x-1)}+\frac{3}{x^2-1}=1\frac{1}{2}.$

26. $\frac{x-1}{x-2}-\frac{x-3}{x-4}+\frac{1}{4}=0.$

27. $\frac{2}{3}+\frac{1}{x+4}+\frac{1}{4x+3}=0.$

28. $\frac{5}{x+1}+\frac{6}{x+2}=\frac{16}{x+4}.$

29. $\frac{7+3x}{4}+\frac{x+9}{x-9}=x+9.$

30. $\frac{1}{3}(x+5)+\frac{2}{5}(x+4)=\frac{1}{x}(x+3).$

PROBLEMS LEADING TO QUADRATIC EQUATIONS.

62. I. The sum of two numbers is 18, and the sum of their squares is 170. Find the numbers.

$$x = \text{the larger number.}$$
$$18 - x = \text{the smaller number.}$$
$$x^2 + (18 - x)^2 = 170.$$

Solving this equation, $x = 11$ or 7.

Then $18 - x = 7$ or 11. *Ans.* 7 and 11.

II. A man bought a quantity of flour for \$105. If he had bought 6 more barrels for the same money, he would have paid \$2 less per barrel. How many barrels did he buy?

$$x = \text{the number of barrels he bought.}$$
$$\frac{105}{x} = \text{the number of dollars paid for one barrel.}$$
$$x + 6 = \text{the number of barrels at the second price.}$$
$$\frac{105}{x+6} = \text{the number of dollars for one barrel at the second price.}$$
$$\frac{105}{x} - 2 = \frac{105}{x+6}.$$

Solving this equation, $x = 15$ or -21. *Ans.* 15 barrels.

NOTE 1. The negative root -21 is not applicable to the problem. Only those values of x which satisfy all the conditions of the problem should be retained as answers.

NOTE 2. If in the problem "6 more" and "\$2 less" should be changed to "6 less" and "\$2 more" respectively, the answer would be 21 barrels. In many cases it is possible to change the wording of a problem so as to form a similar problem whose answer is the absolute value of the negative root of the equation.

EXAMPLES.

1. What number is 12 less than its square?

2. If 24 is added to the square of a certain number, the result is twelve times the number itself. What is the number?

3. Find a number such that nine times the number is greater by 4 than twice the square of the number.

4. Divide 24 into two parts such that their product shall be 135.

5. Divide 20 into two parts such that their product added to the sum of their squares shall give 316.

6. The sum of two numbers is 16, and their product is 55. What are the numbers?

7. The difference between two numbers is 3, and the sum of their squares is 117. What are the numbers?

8. The sum of the squares of two consecutive numbers is 265. What are the numbers?

9. Find three consecutive numbers whose sum is equal to one-half the product of the last two.

10. The square of the sum of two consecutive numbers is greater by 4 than the sum of their squares. Find the numbers.

11. The denominator of a fraction is greater by 2 than its numerator. If both numerator and denominator are increased by 3, the fraction is increased by $\frac{1}{18}$. Find the fraction.

12. The sum of two fractions is $\frac{43}{24}$. The denominator of each fraction is greater by 4 than its numerator, and the numerator of the second fraction is greater by 4 than the numerator of the first fraction. Find the fractions.

13. A number consists of two digits, one of which is the square of the other. If 18 is subtracted from the number, the order of the digits is reversed. Find the number.

14. A number consisting of two digits is equal to three times the product of the digits, and the units' digit is greater by 2 than the tens' digit. Find the number.

15. The length of a room exceeds its breadth by 7 feet, and the area is 330 square feet. Find the length and the breadth of the room.

16. The length of a room exceeds its breadth by 6 feet. If its length is increased by 8 feet, and its breadth by 9 feet, the area is doubled. Find the length and the breadth of the room.

17. A man divided 96 cents among a certain number of boys. If each boy had received 4 cents less, he would have received as many cents as there were boys. Find the number of boys.

18. A man bought some sheep for $120. If the price had been $2 per head less he could have bought three more for the same money. How many sheep did he buy?

19. A man hires a piece of land for $270. He uses 15 acres himself, and lets the remainder for $3 an acre more than he pays for it, receiving just enough to pay for the whole. How many acres does he hire?

20. A man travelled 36 miles in a certain number of hours. If he had taken 3 hours more for the journey, his rate would have been one mile an hour less. Find his rate of travelling.

21. A can do a piece of work in 16 days less time than B, and both together can do it in 15 days. How long will it take each to do it working alone?

22. A man sold a horse for $144 and gained a per cent equal to the number of dollars which the horse cost. Find the cost of the horse.